Example: Vec = category of finite dimensional vector spaces over $\mathbb{k}$

It has operation $V, W \mapsto V \otimes W$

It is easy to impose symmetries

G – (semi) group

consider vector spaces $V : G \curvearrowright V$

If $G \curvearrowright V$, $G \curvearrowright W$ then $G \curvearrowright V \otimes W$

$g(v \otimes w) = gv \otimes gw$

We get a category Rep G.

Q: Can you recover G from representations of G?

Ans: NO Some groups have no non-trivial representations

A: YES for locally compact abelian groups (Pontryagin)

A: YES for compact groups (Tannaka – Krein)

Plan for today:

1. What kind of object is Rep(G)?
2. Use what we observe as definitions.
3. Illustrate in examples (pointed categories)

Rep(G): it is a category

$\mathbb{k}$-linear, abelian

If $\mathbb{k} = \mathbb{C}$ and G is a compact group

$\Rightarrow$ Rep(G) is semisimple

Properties of semisimple categories: upto equivalence
1) Semisimple category is determined by the number of isomorphism classes of simple objects (need $\Bbbk = \overline{\Bbbk}$).

2) $\Bbbk$-linear Functors between semisimple categories are determined (upto natural iso.) by their values on simple objects

(From here on, all functors are $\Bbbk$-linear)

Ex: $G = C_3$ — cyclic group of order 3
$H = S_3$ — symmetric group

$\mathrm{Rep}(G) \simeq \mathrm{Rep}(H)$ as $\Bbbk$-linear categories $\Leftarrow$ both have 3 irreps

So, we need more structure
We have tensor product bifunctor
$$\mathrm{Rep}(G) \times \mathrm{Rep}(G) \longrightarrow \mathrm{Rep}(G)$$

We want to determine how this functor acts on simple objects because bifunctor is determined by its values on simple objects.

To each irrep, $\exists$ character χ_i
$$\chi_i \chi_j = \sum \underbrace{a_k^{ij}}_{\in \mathbb{Z}_{\geq 0}} \chi_k$$

a_k^{ij} tell everything about the tensor bifunctor.

- a_k^{ij} are determined completely by the character table of G.

- C_3 & S_3 have different character table

- But D_8 and Q_8 have the same character table.

$$F: \mathrm{Rep}(D_8) \xrightarrow{\sim} \mathrm{Rep}(Q_8)$$ equivalence compatible with $\otimes$

$$F(X \otimes Y) = F(X) \otimes F(Y)$$

- To differentiate these, need more structure

$\otimes$ is associative

<u>Key point</u>: this is a structure!

$$a : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$$

isomorphism of functors

$$a((v \otimes w) \otimes u) = v \otimes (w \otimes u)$$

FACT: (later) there is no equivalence $\mathrm{Rep}(D_8)$ and $\mathrm{Rep}(Q_8)$ which is compatible with both $\otimes$ and a ⇝ (associativity isomorphism).

This structure is still not enough

<u>Ex</u> (Etingof-Gelaki) There are G, H s.t. $\mathrm{Rep}(G) \xrightarrow{\sim} \mathrm{Rep}(H)$ compatible with both $\otimes$ and a.

Even more structure:

Commutativity

$$c: X \otimes Y \longrightarrow Y \otimes X$$
$$c(v \otimes w) = w \otimes v$$

Fiber functor: $\quad \mathrm{Rep}(G) \longrightarrow \mathrm{Vec} \quad$ $\otimes$ functor

Unit object: $\quad \mathbb{1} = \mathbb{k}$ with trivial G-action

$$\mathbb{1} \otimes X \cong X$$

Tannaka-Krein duality (Tannaka, Krein, Saavendra Rivano, Deligne-Milne)

Group G can be uniquely recovered from $\mathrm{Rep}(G)$ together with $\otimes$, a, c, fiber functor.

Moreover, fiber functor is uniquely determined by $\otimes$, a, c.

Definitions:

$\mathcal{C}$ = $\mathbb{k}$-linear category

$\otimes: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ $\quad$ $\mathbb{k}$-bilinear functor

Associativity isomorphism

$$a: (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$$

self compatible $\quad$ isomorphism of functors

$$\begin{tikzcd}
((X \otimes Y) \otimes Z) \otimes T \arrow[rr] \arrow[d] & & (X \otimes (Y \otimes Z)) \otimes T \arrow[d] \\
(X \otimes Y) \otimes (Z \otimes T) \arrow[dr] \arrow[rr, phantom, "\circlearrowright"] & & X \otimes ((Y \otimes Z) \otimes T) \arrow[dl] \\
& X \otimes (Y \otimes (Z \otimes T)) &
\end{tikzcd}$$

Pentagon axiom

we want this diagram to be commutative

<u>MacLane's Coherence theorem</u>: If Pentagon axiom holds, all diagrams with assiciativity constraint maps commute.

<u>Defn</u>: $\mathcal{C}$ is a semigroup category if it is equipped with $\otimes$, a, where a satisfies the Pentagon axiom.

<u>Unit object</u>: $1 \in \mathcal{C}$, $p : 1 \otimes 1 \xrightarrow{\sim} 1$ isomorphism and

$$\left.\begin{array}{l} X \to 1 \otimes X \\ X \to 1 \otimes X \end{array}\right\} \text{equivalence } \mathcal{C} \to \mathcal{C}$$

(implies $X \xrightarrow{\sim} X \otimes 1 \xrightarrow{\sim} 1 \otimes X$)

<u>Defn</u>: A monoidal category is $(\mathcal{C}, \otimes, a, 1, p)$

satisfies pentagon; unit object

"Tensor category = monoidal category s.t. $\mathcal{C}$ is $\mathbb{k}$-linear, $\otimes$ is $\mathbb{k}$-bilinear"

<u>Ex</u>: G - (semi) group finite

$\rightsquigarrow$

Vec_G = finite dim. vector spaces graded by G

(elements $V = \bigoplus_{g \in G} V_g$)

(this s.s. cat is completely determined by the # of elements in G)

Simple objects:

δ_g: 1-dim'l vector space living in degree g.

Tensor product: $\delta_g \otimes \delta_h \simeq \delta_{gh}$

Another way of saying this is

$$\left(\bigoplus_{g\in G} V_g\right) \otimes \left(\bigoplus_{h\in G} W_h\right) = \bigoplus_{k\in G}\left(\bigoplus_{gh=k} V_g \otimes W_h\right)$$

There is an obvious choice for a.
What about other choices $\tilde{a}$?
To determine it, we want to know

$$\begin{array}{ccc}
\tilde{a} : (\delta_g \otimes \delta_h) \otimes \delta_k & \xrightarrow{\sim} & \delta_g \otimes (\delta_h \otimes \delta_k) \\
\wr\| & & \wr\| \\
\delta_{ghk} & \xrightarrow{\omega(g,h,k)} & \delta_{ghk}
\end{array}$$

Both of these are 1 dim vector spaces
$\therefore$ $\tilde{a}$ is a scalar $\omega(g,h,k) \in \mathbb{k}^*$

$\tilde{a}$ has to satisfy the pentagon axiom.
(check only for simple objects)
It is equivalent to

$$\omega(g_1 g_2, g_3, g_4)\, \omega(g_1, g_2, g_3 g_4)$$
$$= \omega(g_1, g_2, g_3)\, \omega(g_1, g_2 g_3, g_4)\, \omega(g_2, g_3, g_4)$$

ω is a 3-cocycle for G with coefficients in $\mathbb{k}^*$.

Such 3-cocycles do exist.

Ex: Take $G = C_2 = \langle b \rangle$

& define $\omega(g_1, g_2, g_3) = \begin{cases} -1 & \text{if } g_1 = g_2 = g_3 = b \\ 1 & \text{otherwise} \end{cases}$

Exercise: Prove that ω is a 3-cocycle for C_2.

- Choose arbitrary $\mu : G \times G \to \Bbbk^*$
  $\rightsquigarrow \omega(g_1, g_2, g_3) = \dfrac{\mu(g_1 g_2, g_3)\, \mu(g_1, g_2)}{\mu(g_1, g_2 g_3)\, \mu(g_2, g_3)}$

(every coboundary is a cocycle)

Q When two choices of $\tilde{a}$ (or ω) are the "same"?

$\rightsquigarrow$ Notion of tensor functor

Take $\mathcal{C}, \mathcal{D}$ tensor categories

$F : \mathcal{C} \to \mathcal{D}$ functor & want

$\sigma : F(X \otimes Y) \xrightarrow{\sim} F(X) \otimes F(Y)$

want F to be compatible with a.

(*)
$$\begin{tikzcd}
F((X\otimes Y)\otimes Z) \arrow[rr, "F(a)"] \arrow[d, "\sigma"'] & & F(X\otimes(Y\otimes Z)) \arrow[d, "\sigma"] \\
F(X\otimes Y)\otimes F(Z) \arrow[d, "\sigma"'] & \circlearrowleft & F(X)\otimes F(Y\otimes Z) \arrow[d, "\sigma"] \\
(F(X)\otimes F(Y))\otimes F(Z) \arrow[rr, "a"'] & & F(X)\otimes(F(Y)\otimes F(Z))
\end{tikzcd}$$

want this to be commutative.
(There is some condition on unit object too)

Vec_G^{ω} = G-graded vector space with a twisted with ω

$$F : \mathrm{Vec}_G^{\omega} \xrightarrow{\sim} \mathrm{Vec}_H^{\omega'}$$

$$F(\delta_g) = \delta_{\varphi(g)}$$

existence of $\sigma : F(X \otimes Y) \xrightarrow{\sim} F(X) \otimes F(Y)$

$\Rightarrow \varphi$ is an isomorphism of groups

In this case,

$$\sigma : \quad F(\delta_g \otimes \delta_h) \xrightarrow{\sim} F(\delta_g) \otimes F(\delta_h)$$

$$\delta_{\varphi(gh)} \overset{\mu(g,h)}{\dashrightarrow} \delta_{\varphi(g)} \otimes \delta_{\varphi(h)}$$

$\therefore$ σ is specified by choice of a $\mu(g,h) \in \Bbbk^{\times}$

Now, commutativity of ⊛ on last page is equivalent to

$\Longleftrightarrow$

$$\omega(g,h,l)\, \mu(gh,l)\, \mu(g,h) = \omega'(\varphi(g), \varphi(h), \varphi(l))\, \mu(g,hl)\, \mu(h,l)$$

$\Longleftrightarrow$

$$\varphi^{*}\omega' = \omega \quad \text{modulo coboundaries}$$

Thus Vec_G^{ω} upto tensor equivalence are classified by $H^3(G)$