



Diagrammatic methods

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Quantum Symmetries
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Topics

- "Generators mod relations" presentations
of pivotal categories / planar algebras
– an example: the Fibonacci category
- Temperley-Lieb-Jones categories
– Jones-Wenzl idempotents
– an example: the D_{2n} categories
- algebra objects, and skein theory for module categories
- (... diagrams for Drinfeld centres, subfactor planar algebras,
string diagrams for higher categories...?)

Diagrams and quantum symmetries

Victor's talks have given you an algebraic perspective on fusion categories.

$\text{Rep}G$ is a 'prototypical' fusion category, and it's often helpful to think of a general fusion category as a 'noncommutative' or 'quantum' generalisation of a finite symmetry group.

Noah's talks this week will give a rather different perspective, via the cobordism hypothesis:

"Local topological field theories are classified by their value on a point, which is a certain sort of nice higher category".

The essential idea is that local TFTs assign a set of "fields" to each manifold, with rules for gluing together fields.

A higher category just describes the local part, when the manifolds are balls.

Examples

- $Z(M^n) = \mathbb{C}\{\text{principal } G\text{-bundles on } M\}$

\rightsquigarrow the corresponding higher category is $\text{Vec } G$ as an n -category
 ("Dijkgraaf-Witten theory")

- $Z(\text{torus}) = \mathbb{C}\{\text{torus with Wilson lines}\} / \left(\begin{array}{c} \text{Y} = \alpha \\ \text{X} = \beta \end{array} \right) \cup \cap$

\rightsquigarrow the corresponding 2-category is the fusion category Fib .

Perhaps suprisingly every local TFT/higher category
can be described in terms of "string diagrams"
(with perhaps complicated bbds and local relations)
on manifolds/balls.

My lecture series takes this idea seriously, and develops
some of the theory of quantum symmetries (fusion categories, etc)
purely diagrammatically.

Let's suppose we have some skein theory (i.e. local linear relations)

for planar, unoriented, trivalent graphs. (Later: "skein theory" = "pivot category" = "planar algebra")

Let's write \mathcal{P}_n for the space of (formal linear combos of) planar trivalent graphs with n boundary points, modulo these relations.

Let's assume $\mathcal{P}_0 = \mathbb{C}\{\bigcirc\}$, $\mathcal{P}_1 = \mathbb{C}\{\}$, $\mathcal{P}_2 = \mathbb{C}\{\bigcirc/\bigcirc\}$, $\mathcal{P}_3 = \mathbb{C}\{\bigcirc\}$

and $\dim \mathcal{P}_4 \leq 2$

What can we say?

In \mathcal{P}_4 , we have (at least!) the diagrams



Exercise derive a contradiction if $()()$ and \cup are linearly dependent!

We must have a relation of the form

$$\alpha \text{) (} + \beta \text{) } + \text{) } = \text{zero}$$

and so:

$$\alpha \text{) } + \beta \text{) } + \text{) } = 0$$

$$\alpha \text{) } + \beta \text{) } + \text{) } = 0$$

$$\alpha \text{) } + \beta \text{) } + \text{) } = 0$$

Now $\text{) } = \text{zero}$, and $\text{) } = b \text{ / }$ for some b ,

$$\text{so } \alpha + \beta d = \alpha d + \beta + b = \alpha + b = 0.$$

$$\text{Then } d = \frac{1 \pm \sqrt{5}}{2}, \quad \alpha = -b, \quad \beta = -b \frac{1 \pm \sqrt{5}}{2} = \frac{b}{d}$$

$$\text{) (} = \text{) } + \frac{1}{d} \text{) }$$

(and we may as well renormalize the vertex so $b=1$.)

This relation "looks really powerful" and suggests there's perhaps at most one ^{*} example satisfying these conditions!

* (two: $d = \frac{1 \pm \sqrt{5}}{2}$)

Why?

Claim 1 $\Upsilon = \left(-\frac{1}{d} \cup \cap \right)$ allows us to evaluate all closed diagrams

Proof In a closed diagram, every Υ must be adjacent to another Υ ! The relation lets us replace that part of the diagram with a linear combination of strictly simpler diagrams.


Example


$$\begin{aligned}
 \triangle &= \bigcirc - \frac{1}{d} \bigcirc \\
 &= \left[-\frac{1}{d} \bigcirc - \frac{1}{d} \bigcirc + \frac{1}{d^2} \bigcirc \right] \\
 &= d - d - d + d^{-1} = 1
 \end{aligned}$$

Such a skein theory (i.e. with $\dim P_0 \leq 1$) is called evaluable.

(Equivalently, in a semisimple category, the tensor unit is simple.)

A general fact: every ideal in an evaluable skein theory is contained in the negligible ideal.

Defn  is negligible if every closed diagram built using f is zero.

Proof Say  $\in I$, and non-negligible.

Then there is some  so  $\neq 0$,

so the empty diagram is in I , and hence I is the entire skein theory.

If we've discovered some relations that let us prove evaluability,
form the free skein theory given by those generators and relations.

In our example, let Q be

$$\left\langle \text{Y} \mid \bigcirc = \frac{4\sqrt{5}}{2}, \quad \parallel = \text{Y} + \frac{1}{d} \cup \right\rangle$$

It's evaluable, and there's a "functor"

$$F: Q \longrightarrow P$$

(Later, when we interpret P and Q as categories, this will be surjective on objects and morphisms)

Thus P is completely determined by the kernel of this
functor, which is an ideal, and
hence a sub-ideal of Q 's unique maximal ideal,
the negligibles.


If we add the assumption that \mathcal{P} is nondegenerate,
we're done: \mathcal{P} must be \mathbb{Q}/\mathcal{N} .

(Although we may not be satisfied with this description
if we don't know a good description of \mathcal{N} .)

In fact, in our example $\mathcal{N} \neq \{0\}$!

Observe  = $\int \text{---} - \frac{1}{d} \text{---} = \text{zero}$,

and since in every closed diagram  appears next to another vertex,

 $\in \mathcal{N}$. In fact $\mathcal{N} = \langle \text{---} \rangle$, but we're not quite ready
to prove this.

We potentially have a bigger problem...

what if Q is actually zero!?

(This could happen if the relations let us evaluate a closed diagram in two different ways. This would say $\dim Q_0 = 0$ and then $\dim Q_n = 0$ for all n .)

Three solutions

(A) Directly show Q is nonzero, using 'confluence'.

This is usually really hard - I don't even know how to do this case!

(B) Find Q (or a non-zero quotient) "elsewhere in mathematics".

For this example, we're in luck: $Q/\mathcal{N} \cong \text{Rep } U_{\xi_{10}} \underline{SO}_3$. (Or via Temperley-Lieb or the chromatic polynomial.)

(C) Attempt a "general purpose" construction, using

graph planar algebras / the regular representation.
Hopefully we'll have time later in the week.

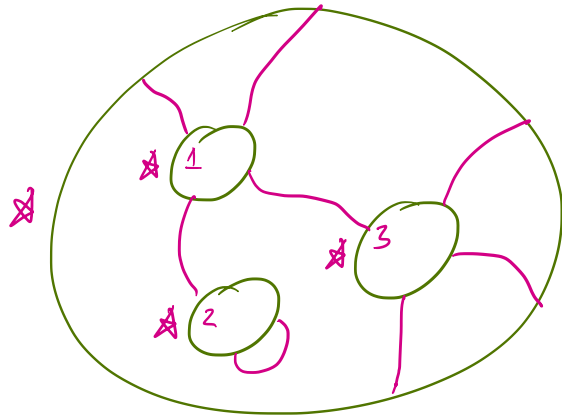
One useful formalisation of 2-dimensional string diagrams

is as a planar algebra.

(due to V. Jones, arXiv:9909027)

A planar algebra \mathcal{P} consists of:

- a collection of vector spaces \mathcal{P}_n , for $n \in \mathbb{N}$.
- for each "spaghetti and meatballs" diagram



a linear map $\mathcal{P}_4 \otimes \mathcal{P}_3 \otimes \mathcal{P}_4 \rightarrow \mathcal{P}_5$

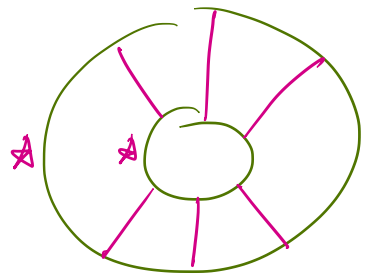
(from the tensor product of the vector spaces associated to the inner circles to the vector space associated to the outer circle).

- such that

- gluing one diagram inside another is compatible with composing the corresponding linear maps:

$$P \left(\text{Diagram 1} \right) = P \left(\text{Diagram 2} \right) \circ \left(P \left(\text{Diagram 3} \right) \otimes \text{id}_{P_3} \right)$$

- the 'radial' diagrams



act by the identity

- two diagrams which are isotopic rel boundary act by the same linear map.

Example Temperley-Lieb-Jones forms a planar algebra:

$$\text{TLJ}_n = \mathbb{C} \{ \text{TLJ diagrams with } n \text{ bdy points} \}$$

(the "vegetarian" spaghetti
and meatballs
diagrams)

$$0 = \delta \in \mathbb{C}$$

- Exercises
- define a morphism of planar algebras
 - explain why there is a morphism $\text{TLJ} \rightarrow \mathcal{P}$
for any planar algebra \mathcal{P} with $\dim \mathcal{P}_0 = 1$.
 - discover and prove a formula for $\dim \text{TLJ}_n$
 - invent some generalisations of this definition!
 - what if I wanted to allow oriented spaghetti?
 - or a Colombian planar algebra:



Another perspective on string diagrams starts with the traditional language of monoidal categories.

A monoidal string diagram for a ^{strict} monoidal category \mathcal{C}

is a planar diagram consisting of

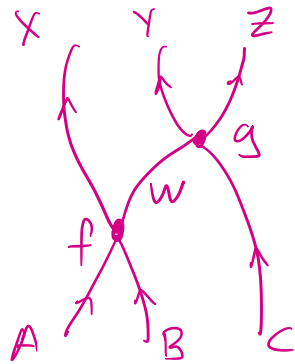
- oriented strings, which point up the page,

labelled by objects of \mathcal{C}

- "vertices"/"boxes"/"coupons"

with incoming strings and outgoing strings,

labelled by appropriate morphisms from \mathcal{C}



$$f: A \otimes B \rightarrow X \otimes W$$

$$g: W \otimes C \rightarrow Y \otimes Z$$

We can represent any morphism in \mathcal{C} as a string diagram, representing composition using "vertical stacking" and tensor product using "horizontal juxtaposition".

$$\begin{array}{c} \text{---} \\ \bullet \\ \text{---} \\ \bullet \\ \text{---} \end{array} \begin{array}{l} f \\ g \end{array} = \begin{array}{c} \text{---} \\ \bullet \\ \text{---} \end{array} \begin{array}{l} g \circ f \end{array}, \quad \begin{array}{c} \text{---} \\ \bullet \\ \text{---} \end{array} \begin{array}{l} f \\ g \end{array} = \begin{array}{c} \text{---} \\ \bullet \\ \text{---} \end{array} \begin{array}{l} f \otimes g \end{array}$$

Theorem If two monoidal string diagrams are isotopic (through an isotopy which neither introduces critical points in strings nor rotates vertices) then the corresponding morphisms are equal.

Exercise Check $\begin{array}{c} \text{---} \\ \bullet \\ \text{---} \end{array} \begin{array}{l} f \\ g \end{array} = \begin{array}{c} \text{---} \\ \bullet \\ \text{---} \end{array} \begin{array}{l} f \\ g \end{array}$ in any monoidal category.

How many axioms did you use?

Definition a monoidal string diagram category \mathcal{C} consists of:

- a set \mathcal{L} of edge labels
- for each pair of words $w_{in}, w_{out} \in \mathcal{L}$,
a set $\mathcal{L}(w_{in} \rightarrow w_{out})$ of vertex labels
- for each pair of words $w_{in}, w_{out} \in \mathcal{L}$,
a subspace $\mathcal{U}(w_{in} \rightarrow w_{out})$ of the (formal linear combos of)
monoidal string diagrams labelled using \mathcal{L} with
incoming boundary w_{in} and outgoing boundary w_{out}
- such that \mathcal{U} forms an ideal under vertical stacking
and horizontal juxtaposition
- and if two string diagrams f and g are
monoidally isotopic rel ∂ , then $f - g \in \mathcal{U}$.

From this we can build a monoidal category $\hat{\mathcal{C}}$, by

$$\text{Obj } \hat{\mathcal{C}} = \mathcal{L}$$

$$\hat{\mathcal{C}}(X \rightarrow Y) = \frac{\mathcal{C}\{\mathcal{L}\text{-labelled string diagrams from } X \text{ to } Y\}}{\mathcal{U}(X \rightarrow Y)}$$

Theorem every monoidal category is monoidally equivalent to one of this form.

Sketch Define $\mathcal{L} = \text{Obj } \mathcal{C}$.

Define $\mathcal{L}(X \rightarrow Y) = \mathcal{C}(X \rightarrow Y)$ (i.e. the vertex label set is just all morphisms)

We have an 'evaluation map' taking string diagrams to morphisms in \mathcal{C} .

Define $\mathcal{U} = \ker(\text{eval})$.

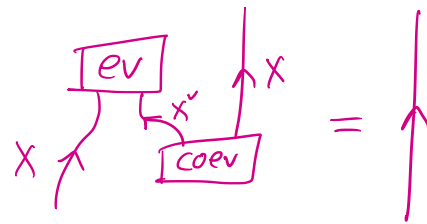
Recall a monoidal category is rigid if

- for every object X there is dual object X^\vee

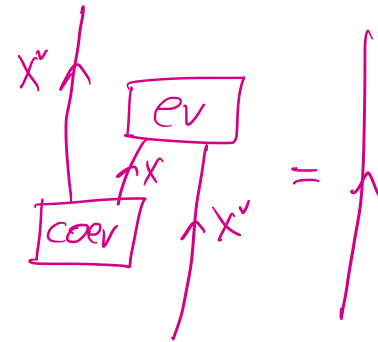
and maps $ev: X \otimes X^\vee \rightarrow \mathbb{1}$

$coev: \mathbb{1} \rightarrow X^\vee \otimes X$

satisfying



and



- and every X has a pre-dual ${}^\vee X$ so $({}^\vee X)^\vee = X$.

In a rigid string diagram strings can go up or down the page,
and must be oriented to the right at any critical points.

Isotopies may introduce or cancel pairs of critical points,
but may not rotate vertices.

Now:

① we interpret rigid string diagrams in \mathcal{C} using

$$\downarrow x \rightsquigarrow \text{id}_{x^\vee}$$

$$x \curvearrowright \rightsquigarrow \text{ev}_x: X \otimes X^\vee \rightarrow \mathbb{1}$$

$$\curvearrowleft x \rightsquigarrow \text{coev}_x: \mathbb{1} \rightarrow X^\vee \otimes X$$

② Theorem two isotopic rigid string diagrams evaluate to the same morphism

③ Theorem Every rigid monoidal category is equivalent to a category of rigid string diagrams.

Finally recall we can assemble $X \mapsto X^\vee$ into a monoidal functor $v: \mathcal{C} \rightarrow \mathcal{C}^{op, \text{map}}$,

defined on morphisms by

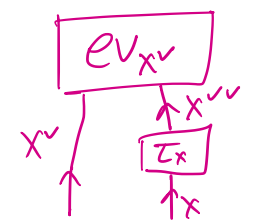
$$\left(\begin{array}{c} \uparrow y \\ \boxed{p} \\ \uparrow x \end{array} \right)^\vee := \begin{array}{c} \downarrow \\ \boxed{p} \\ \uparrow \end{array}$$

(This is a good exercise!)

and a pivotal structure is a choice of monoidal natural isomorphism

$$Z: \text{id}_{\mathcal{C}} \xrightarrow{\sim} v \circ v$$

Now a (pivotal) string diagram has no constraints on the tangencies,
 we interpret $x \psi \rightsquigarrow$

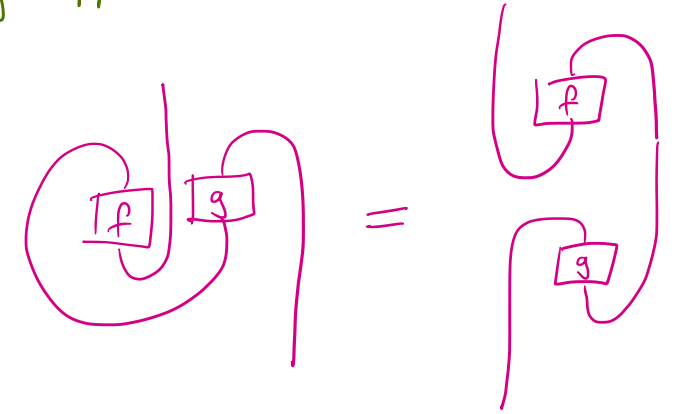


a (pivotal) isotopy may freely rotate vertices (i.e. it's just a general isotopy)

Then (1) Theorem: isotopic string diagrams have the same evaluation

(2) Theorem: every pivotal category is pivotal equivalent to a category of pivotal string diagrams.

Exercise How many applications of an 'algebraic' axiom are required to prove:



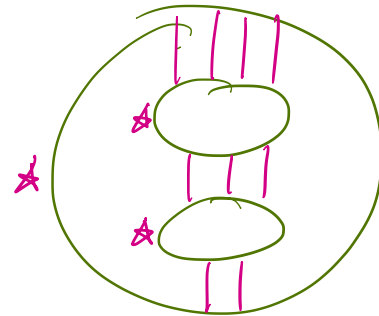
Both planar algebras and pivotal categories are
 'just' the theory of planar diagrams up to planar isotopy.

Given a planar algebra \mathcal{P} , we define a pivotal category $\mathcal{C}_{\mathcal{P}}$

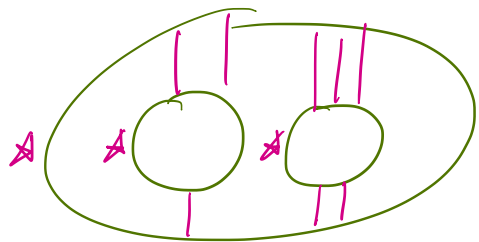
by: $\text{Obj } \mathcal{C}_{\mathcal{P}} = \mathbb{N}$

$\mathcal{C}_{\mathcal{P}}(n \rightarrow m) = \mathcal{P}_{n+m}$

Composition is provided by the tangle



and tensor product is $+$ on objects and provided by



on morphisms.

Exercises:

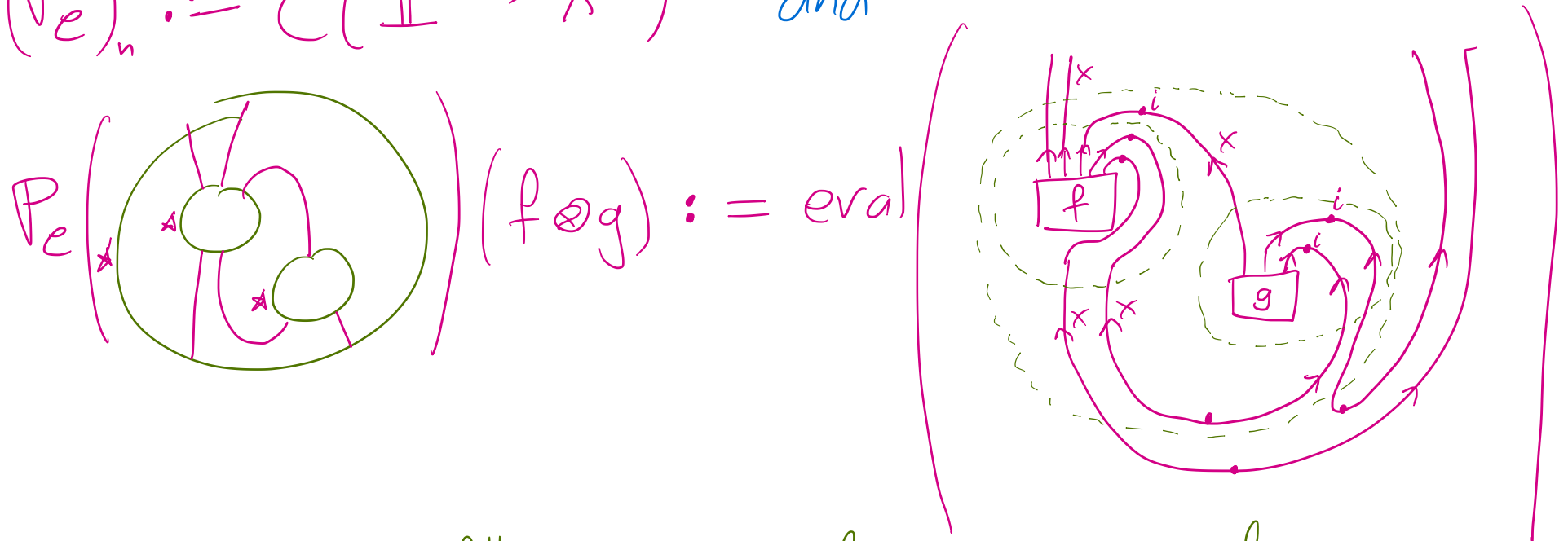
- check this is rigid
- write down the 'obvious' pivotal structure.

Given a pivotal category \mathcal{C} and a choice of an object X which is symmetrically self-dual

(i.e. we have $\downarrow \begin{matrix} X \\ \bullet \\ i \\ X \end{matrix} \uparrow$ and $\downarrow \begin{matrix} \uparrow \\ \bullet \\ i \\ \downarrow \end{matrix} \uparrow = \bullet \begin{matrix} | \\ i \\ | \end{matrix}$)

we define a planar algebra $\mathcal{P}_{\mathcal{C}}$ by

$(\mathcal{P}_{\mathcal{C}})_n := \mathcal{C}(\mathbb{1} \rightarrow X^{\otimes n})$ and



Exercise explain carefully why this satisfies the axioms of a planar algebra!

(In fact this is much easier if we work with the oriented planar algebras you defined earlier; then we don't need to ask that X is symmetrically self-dual.)

Theorem $\mathcal{C}_{\mathcal{P}, X} \underset{\text{Pivotal}}{\cong} \langle X \rangle_{\mathcal{C}}$ (the full subcategory of \mathcal{C} on objects $X^{\otimes n}$)

Theorem $\mathcal{P}_{\mathcal{C}, 1} \underset{\text{P.A.}}{\cong} \mathcal{P}$ (see e.g. arXiv:0810.4186)

At first it feels disconcerting to hear

$$\text{Obj } \mathcal{C}_P = \mathbb{N}$$

What about examples like $\text{Rep } G$, where the objects are representations, and decompose uniquely as direct sums of irreps?

Slogan "objects don't matter"

We recover the objects we expect to see by taking the idempotent completion / Karoubi envelope.

For any 1-category \mathcal{C} ,

$$\text{Obj Kar } \mathcal{C} := \{ (X : \mathcal{C}, p : \mathcal{C}(X \rightarrow X), p^2 = p) \}$$

$$\text{Kar } \mathcal{C}((X, p) \rightarrow (Y, q)) := \{ (f : \mathcal{C}(X \rightarrow Y), p \gg f = f = f \gg q) \}$$

If \mathcal{C} is monoidal/rigid/pivotal/..., so is $\text{Kar } \mathcal{C}$ in a natural way.

We also need to formally adjoin direct sums.

$\text{Mat}(\mathcal{C})$ has objects words in \mathcal{C}

and $\text{Mat}(\mathcal{C})(\bigoplus X_i \rightarrow \bigoplus Y_j) = \left\{ \left(\begin{array}{c} \vdots \\ \dots a_{ij} : \mathcal{C}(X_i \rightarrow Y_j) \dots \\ \vdots \end{array} \right) \right\}$

with composition by matrix multiplication.

(again, when \mathcal{C} is monoidal/rigid/pivotal/etc $\text{Mat}(\mathcal{C})$ inherits this)

Exercise • show $\text{Kar}(\mathcal{C}) \cong \mathcal{C}$

• if \mathcal{C} already has direct sums, show $\text{Mat}(\mathcal{C}) \cong \mathcal{C}$.

$\text{Mat}(\text{Kar}(\mathcal{C}))$ is the "Cauchy completion"

Finally, without having to use abelian categories,
we can say \mathcal{C} is semisimple if

there exist objects X_i so

$$\dim \mathcal{C}(X_i \rightarrow X_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$$

and every object is isomorphic to a direct sum of the X_i 's

(or, more generally, if $\text{Mat}(\text{Kar}(\mathcal{C}))$ satisfies this condition)

Let's return to our stem theory

$$Q = \langle \text{Y} \mid \circ = \frac{H\sqrt{5}}{2}, \parallel = \text{Y} + \frac{1}{d} \cup, \rho = \text{zero} \rangle$$

and study the idempotents.

We have $\mathbb{1} = \square$, and $X = \square \mid$.

Claim every idempotent is a direct sum of copies of $\mathbb{1}$ and X ,
and in particular $X \otimes X \cong \mathbb{1} \oplus X$.

What does $A \cong B \oplus C$ even mean? We have morphisms

$$A \begin{array}{c} \xrightarrow{\pi_1} \\ \xrightarrow{\pi_2} \end{array} \begin{array}{c} B \\ \oplus \\ C \end{array} \begin{array}{c} \xrightarrow{L_1} \\ \xrightarrow{L_2} \end{array} A, \quad \text{so} \quad \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} \gg (L_1 \ L_2) = (\pi_1 \gg L_1 + \pi_2 \gg L_2) = \text{id}_A,$$

$$\text{and } (L_1 \ L_2) \gg \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} = \begin{pmatrix} L_1 \gg \pi_1 & L_2 \gg \pi_1 \\ L_1 \gg \pi_2 & L_2 \gg \pi_2 \end{pmatrix} = \begin{pmatrix} \text{id}_B & 0 \\ 0 & \text{id}_C \end{pmatrix}.$$

We can construct an explicit isomorphism $X \otimes X \cong \mathbb{1} \oplus X$ as follows:

$$\begin{array}{ccccc}
 X \otimes X & \xrightarrow{\quad \cap \quad} & \mathbb{1} & \xrightarrow{\quad \overset{\cap}{\downarrow} \quad} & X \otimes X \\
 & & \oplus & & \\
 & \xrightarrow{\quad \cup \quad} & X & \xrightarrow{\quad \overset{\cup}{\downarrow} \quad} & X \otimes X
 \end{array}$$

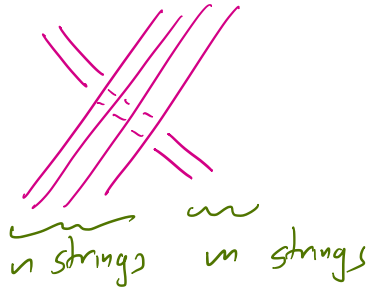
and check: $\overset{\cap}{\downarrow} \circlearrowleft = \circlearrowleft$, the identity on $\mathbb{1}$.

$\overset{\cup}{\downarrow} = |$, the identity on X ,

and $\overset{\cup}{\downarrow} + \overset{\cap}{\downarrow} = |$ is exactly the relation we just discovered!

- Exercises
- explain why there are no maps between $\mathbb{1}$ and X .
 - prove $X^{\otimes n} \cong F_{n-1} \mathbb{1} \oplus F_n X$, where the F_n are Fibonacci numbers
 - argue that every object in the idempotent completion is a subobject of $X^{\otimes n}$ for some n , and hence show the idempotent completion is semisimple.

Exercises • Explain which axiom for a braiding ensures that

if $\beta_{1,1} = \text{X}$, then $\beta_{n,m} =$ 

Thus in Fib a braiding is determined by $r, s: \mathbb{C}$,

$$\text{X} = r \text{) } (+ s \text{) } ($$

• Explain, from the axioms for a braiding, why

$$\text{X} := \text{X}^{-1} = \text{X}$$

and solve for r and s

• Explain why the only remaining conditions are


$$\text{X} = \text{X} \text{ (with a dot on the top string)} \quad \text{and} \quad \text{X} = \text{X} \text{ (with a dot on the bottom string)} \quad \text{and verify these.}$$

D_{2n} categories

(following arXiv:0808.0764)


We've seen already that the Temperley-Lieb-Jones category is

the "simplest possible" pivotal category from the "generators and relations" point of view —

• generators: an unoriented arc 

• relations:  = $q + q^{-1}$

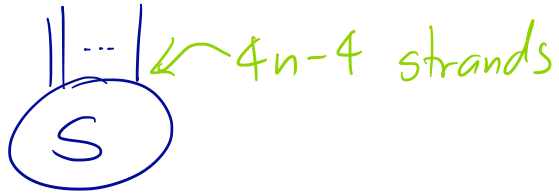
(and, when q is a primitive $(2\lambda+2)$ th root of unity,

 = zero)

We'll now study the "next simplest", with a single generator.

Defn Fix $n \geq 1$, and let $q = \exp\left(\frac{\pi i}{4n-2}\right)$.

Consider the pivotal category / planar algebra / skein theory with generator



modulo the relations

① $\bigcirc = q + q^{-1}$

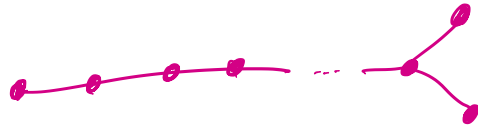
④ $\begin{array}{c} \text{---} \\ \parallel \\ \text{---} \\ \bigcirc \\ \text{---} \\ \parallel \\ \text{---} \end{array} = [2n-1] \cdot \begin{array}{c} \text{---} \\ \parallel \\ \text{---} \\ \boxed{f^{(4n-4)}} \\ \parallel \\ \text{---} \end{array}$

② $\begin{array}{c} \text{---} \\ \parallel \\ \text{---} \\ \bigcirc \\ \text{---} \\ \parallel \\ \text{---} \end{array} = i \begin{array}{c} \text{---} \\ \parallel \\ \text{---} \\ \bigcirc \\ \text{---} \\ \parallel \\ \text{---} \end{array}$

③ $\begin{array}{c} \text{---} \\ \parallel \\ \text{---} \\ \bigcirc \\ \text{---} \\ \parallel \\ \text{---} \end{array} = \text{Zero}$

Theorem

- (A) every closed diagram is a scalar multiple of the identity (the tensor unit is simple)
- (B) the isomorphism classes of simple idempotents are indexed by vertices of the D_{2n} Dynkin diagram



and $\begin{array}{|c|} \hline \dots \\ \hline p \\ \hline \dots \\ \hline \end{array} \Big|$ is isomorphic to the direct sum of the idempotents associated to the vertices adjacent to p in this graph.

To prove either of these theorems we need

$$\boxed{\begin{array}{c} \dots \\ p^{(4n-3)} \\ \dots \end{array}} = \text{zero} \quad \text{and}$$

$$\left(\begin{array}{c} \dots \\ \text{S} \\ \dots \end{array} \right) = \left(\begin{array}{c} \dots \\ \text{S} \\ \dots \end{array} \right)$$

Note $f^{(4n-3)} \in \mathcal{N}$ at $q = \exp\left(\frac{\pi i}{4n-2}\right)$. In fact it is zero as a consequence of the relations:

$$[4n-3] \begin{array}{c} \dots \\ p^{(4n-3)} \\ \dots \end{array} = [4n-3] \begin{array}{c} \dots \\ p^{(4n-4)} \\ \dots \end{array} - [4n-4] \begin{array}{c} \dots \\ p^{(4n-4)} \\ \dots \end{array}$$

$$= \frac{1}{[2n-1]} \begin{array}{c} \dots \\ \text{S} \\ \dots \\ \text{S} \\ \dots \end{array} - \frac{[2]}{[2n-1]^2} \begin{array}{c} \dots \\ \text{S} \\ \dots \\ \text{S} \\ \dots \\ \text{S} \\ \dots \end{array} = \frac{1}{[2n-1]} \begin{array}{c} \dots \\ \text{S} \\ \dots \\ \text{S} \\ \dots \end{array} - \frac{[2]}{[2n-1]} \begin{array}{c} \dots \\ \text{S} \\ \dots \\ \boxed{p^{(4n-4)}} \\ \dots \\ \text{S} \\ \dots \end{array}$$

$$= \text{zero}$$

Then we check each way of adding a cap to the 2nd equation gives zero:

$$\text{Cap on left} = i \text{Cap on second} = \text{Cap on right} \quad (\text{by expanding out the TLJ braiding!})$$

$$\text{Cap on left} = \text{zero} = \text{Cap on right} \quad \text{etc}$$

Now

$$\text{Cap on left} = \text{Cap on second} \left[\frac{1 - p(2n-1)}{1 - p(2n-1)} \right]$$

every term here has a cap

$$= \text{Cap on second} \left[\frac{1 - p(2n-1)}{1 - p(2n-1)} \right] = \text{Cap on right}$$

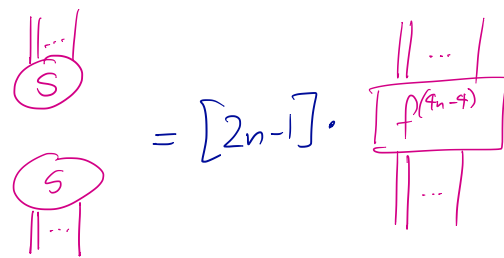
Note, however, that $\text{Cap on left} \neq \text{Cap on right}$!

Now prove theorem (A) — every closed diagram can be evaluated using these relations:

(1) Pick a pair of S 's, and slide them towards each other using the 'half-braiding'



(2) Once they're adjacent, remove them using



(3) Repeating (1), we get down to 0 or 1 S 's.

(4) If there's 1, it must have a cap, so return zero

(5) Otherwise, use $\bigcirc = q + q^{-1}$ to evaluate to a number.

In fact, one can prove this algorithm is independent of choices and respects all the defining relations, giving a direct proof of the consistency of this presentation! (See arXiv:0808.0764)

Algebras

An algebra (in a monoidal category \mathcal{C}) consists of:

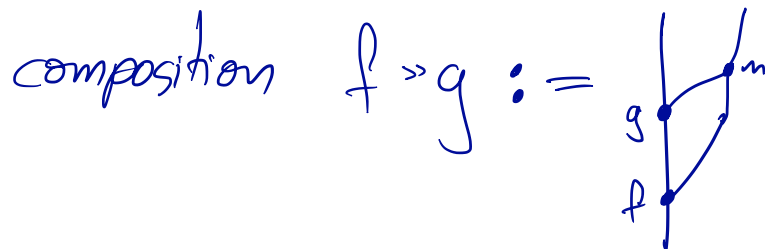


We can define the projective modules for an algebra A entirely diagrammatically.

First define $\text{fmod-}A$, a category, with

- $\text{Obj}(\text{fmod-}A) = \text{Obj}(\mathcal{C})$
- $\text{fmod-}A(X \rightarrow Y) = \mathcal{C}(X \rightarrow YA)$

with identity $1_x := \begin{array}{c} x \quad A \\ \uparrow \\ \uparrow \\ x \end{array}$ and



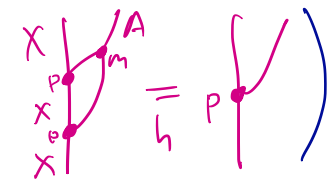
Exercise verify composition is associative!

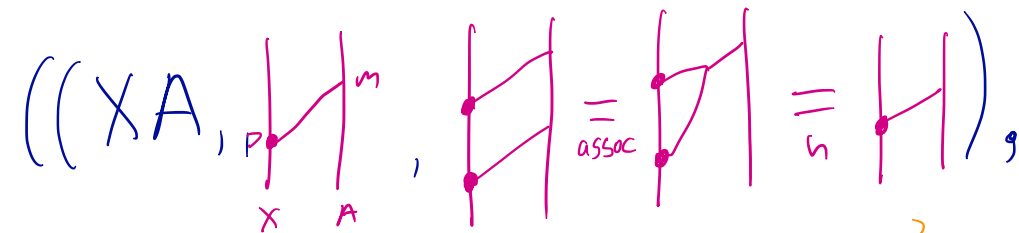
We then define $\text{pmod-}A := \text{Ker}(\text{Fmod-}A)$.

What does this have to do with the usual $\text{mod-}A$, which is defined to be the category of A -module objects internal to \mathcal{C} ?

In general, even in Vec , they disagree — plenty of (honest) algebras have modules which are not projective. We might expect to need A semisimple.

There is a functor, however, $i: \text{pmod-}A \xrightarrow{\text{Ker}} \text{mod-}A$

sending $(X: \mathcal{C}, p: \mathcal{C}(X \rightarrow XA))$ 

to $((XA, \text{diagram})$ 

the underlying object in $\text{Ker } \mathcal{C}$

$\triangleleft := \left(\text{diagram}, \text{Exercise: this is a map in } \text{Ker } \mathcal{C} \right), \text{ Exercise: this map satisfies the axioms for a module object action}$

Exercise define the functor $i: {}_E \text{pmod-} A \rightarrow \text{mod-} A$
core
 at the level of morphisms, and verify functoriality.

Exercise the functor i is fully-faithful.

Theorem If A is a special Frobenius algebra in \mathcal{C} ,
 then the functor $i: {}_E \text{pmod-} A \rightarrow \text{mod-} A$
 is an equivalence of categories.

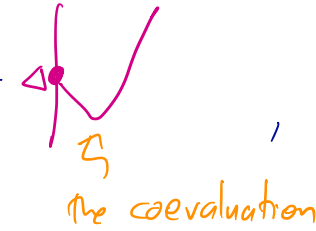
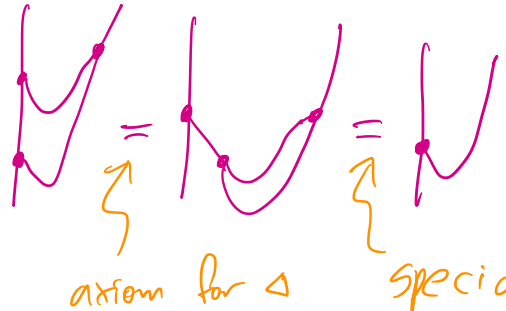
Definition a special Frobenius algebra in a pivotal category \mathcal{C} is a
 symmetrically self-dual algebra further satisfying

$$\text{cap}_m = \text{cup}_m, \quad \text{tr}_m = 1$$

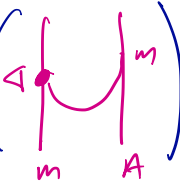
Exercise: how does this
 relate to the usual definition?

Proof We just need to verify i is essentially surjective.

Given $(m, \triangleleft) \in \text{mod-}A$, let's define a projective module

$(X := m, p :=$

 $, h :=$


(for simplicity, let's assume \mathcal{C} is idempotent complete, so we don't need to worry about $\text{Kar } \mathcal{C}$)

Now $i(X, p, h) = \text{Image}(\triangleleft$


which is isomorphic (in $\text{mod-}A$) to the original m ,

via the maps  and .

Exercise: verify this isomorphism, carefully noting when you need to use specialness.

Now, if \mathcal{C} is braided, and the algebra A is commutative, so $\overline{\mathcal{C}} = \mathcal{C}^m$

we can define a monoidal structure on $f\text{mod-}A$:

on objects, the tensor product is the same as in \mathcal{C}

on morphisms, $f \otimes g =$ 

Exercise which axiom for a monoidal category do we need commutativity for?

This then gives a monoidal structure on $p\text{mod-}A = \text{Kar}(f\text{mod-}A)$.

We'd like to describe an intrinsic generators mod relations presentation of $f\text{mod-}A$.

The story so far:

- the D_{2n} categories, with relations

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \\ \cup \\ | \\ \text{---} \\ \cap \\ | \\ \text{---} \end{array} = i \begin{array}{c} \text{---} \\ | \\ \text{---} \\ \cap \\ | \\ \text{---} \end{array}, \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \\ \cap \\ | \\ \text{---} \end{array} = \left\{ \begin{array}{c} \text{---} \\ | \\ \text{---} \\ \boxed{f(1_{n-1})} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} \right.$$

are evaluable, but we haven't yet computed
the minimal idempotents and their tensor products

- the categories $f\text{mod-}A$, $p\text{mod-}A := \text{Ker}(p\text{mod-}A)$, and $\text{mod-}A$,
and a diagrammatic proof that

$$e p\text{mod-}A \cong_{\text{Ker}} \text{mod-}A$$

when A is a special Frobenius algebra.

Theorem Suppose $\mathcal{C} = \langle X_\alpha, r_\beta \rangle$ is a presentation of a braided monoidal category.

Say A is an algebra in \mathcal{C} , and $A \cong \bigoplus P_i$,

where the P_i are minimal projections in \mathcal{C} .

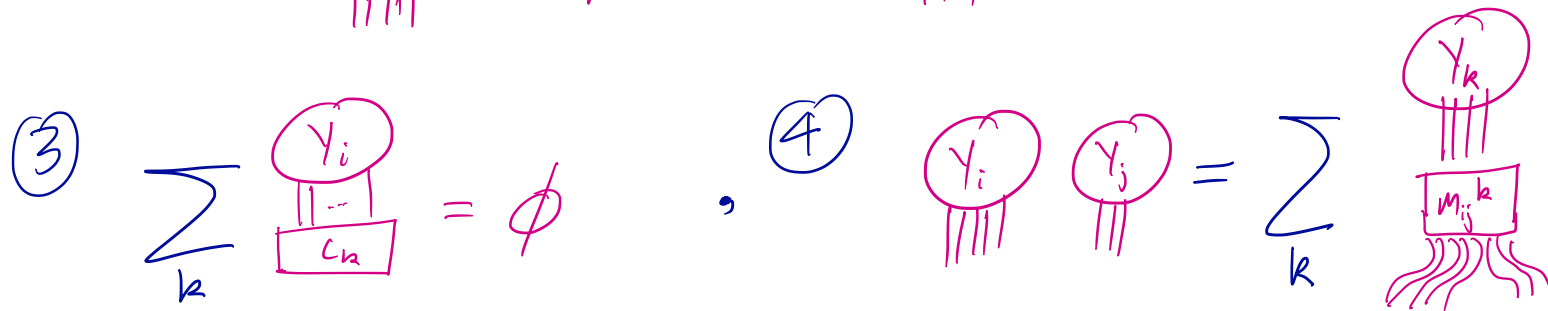
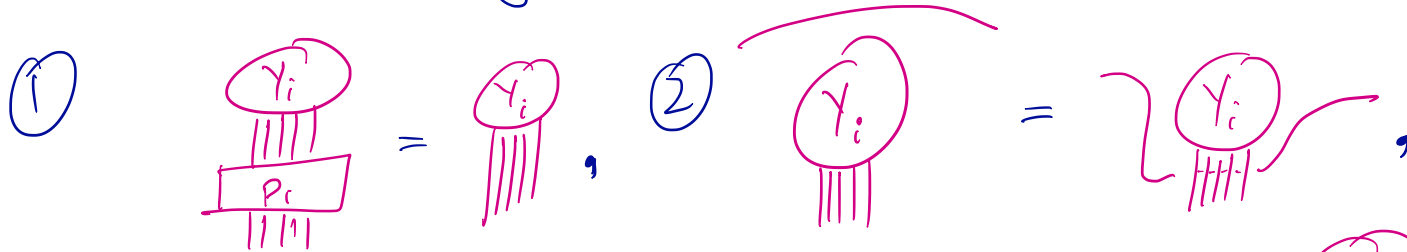
We have $m = \sum_{i,j,k} m_{ij}^k$, where $m_{ij}^k: P_i \otimes P_j \rightarrow P_k$,

and $\iota = \sum \iota_k$, where $\iota_k: \mathbb{1}_{\mathcal{C}} \rightarrow P_i$

Then $\text{fmod-}A$ has a presentation \mathcal{P} , with

generators $\{X_\alpha, Y_i\}$ with one Y_i for each P_i , and

relations $\{r_\beta\}$ along with



Proof sketch We'll define functors $F: \text{fmod-}A \rightleftarrows \mathcal{P}: G$
 (where \mathcal{P} is the category given by the presentation).

$$F\left(\begin{array}{c} y \\ | \\ \bullet \\ | \\ x \end{array} \begin{array}{c} / \\ \backslash \\ A \end{array}\right) := \sum_i \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \begin{array}{c} / \\ \backslash \\ A \end{array} \begin{array}{c} y_i \end{array}$$

(here, $\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \begin{array}{c} / \\ \backslash \\ A \end{array}$ denotes the projection from A to the summand P_i , followed by Y_i .)

$$G\left(\begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ | \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ | \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \end{array}\right) := \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ | \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ | \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \end{array}$$

(The diagram on the right shows a more complex structure with labels P_i , A , and m .)

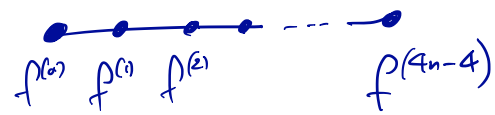
Exercises

- make this as precise as you like!
- why are F and G monoidal functors?
- why are F and G inverses?
- see that it all works if \mathcal{C} is not braided but A is a commutative algebra in $\mathcal{Z}(\mathcal{C})$.
- when is $\text{fmod-}A$ pivotal?

How does this explain the D_{2n} skein theory?

What are the algebras in TL at $q = \exp\left(\frac{\pi i}{4n-4}\right)$?

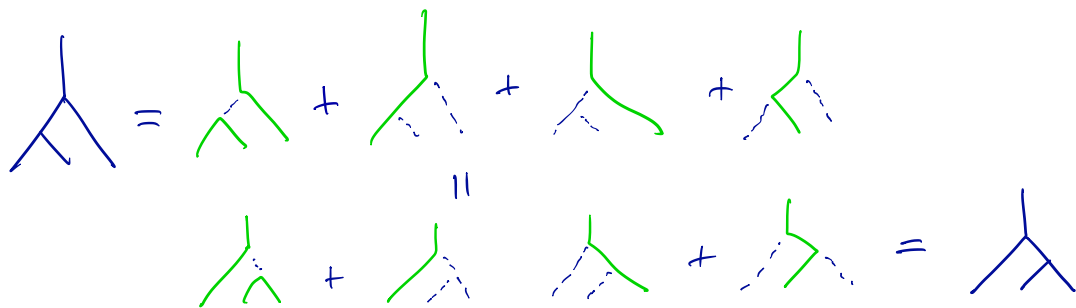
This is in A_{4n-3} , with principal graph



Claim $A := f^{(0)} \oplus f^{(4n-4)}$ has an algebra structure given by:

	$f^{(0)}$	\oplus	$f^{(4n-4)}$	$= A$
$f^{(0)} \otimes f^{(0)}$			0	
$f^{(0)} \otimes f^{(4n-4)}$	0			
$f^{(4n-4)} \otimes f^{(0)}$	0			
$f^{(4n-4)} \otimes f^{(4n-4)}$			0	

Sketch:



where the interesting calculation is



which only holds at $q = \exp\left(\frac{\pi i}{4n-4}\right)$

Aside: in fact, we know all the algebra objects in TLJ at different roots of unity (Ostrik, arXiv:0111139, but also the earlier subfactor literature.)

Algebra objects. The simple algebra objects A in the monoidal categories $\mathcal{C}_{k,m}$ are as follows.

A	when	$A\text{-mod}$	commutative in $\mathcal{C}_{k,\ell}^{\text{br}}$
$1 \oplus f^{(k)}$	k even	$D_{\frac{k}{2}+2}$	if $k \equiv 0 \pmod{4}$
$1 \oplus f^{(k)}$	k, m both odd	$T_{\frac{k+1}{2}}$	if $k\ell \equiv 3 \pmod{4}$
$1 \oplus f^{(6)}$	$k = 10$	E_6	yes
$1 \oplus f^{(8)} \oplus f^{(16)}$	$k = 16$	E_7	no
$1 \oplus f^{(10)} \oplus f^{(18)} \oplus f^{(28)}$	$k = 28$	E_8	yes

TLJ at $q = \frac{m+1}{k+2}$, with principal graph A_{k+1} .

See [Ost03a] for details. For details on the categories $A\text{-mod}$ (in subfactor language) see [BN91; Izu91; Izu94; Jon01; Kaw95; Xu98]. When A is commutative the category $A\text{-mod}$ has the structure of a fusion category. For details on the skein theory of these fusion categories see [Big10; MPS10]

(From "A field guide to categories with A_n fusion rules", arXiv:1710.07362)

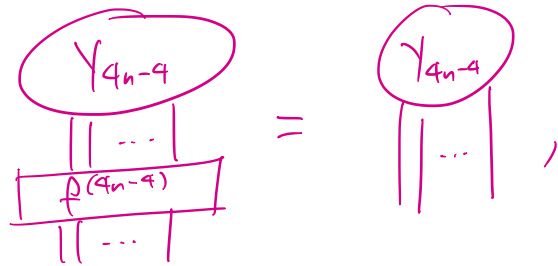
What is the skein theory for $f_{\text{mod}} - (f^{(0)} \oplus f^{(4n-4)})$?

There are two new generators corresponding to the simple summands of the algebra.



They satisfy:

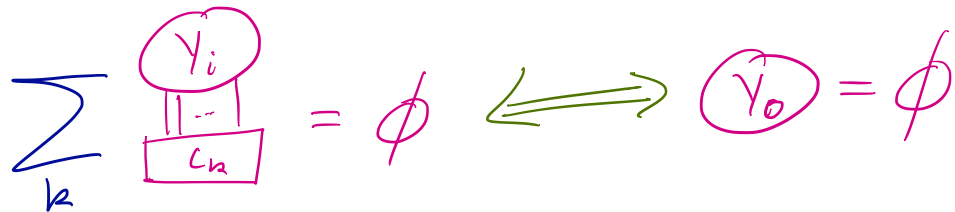
①:



②:



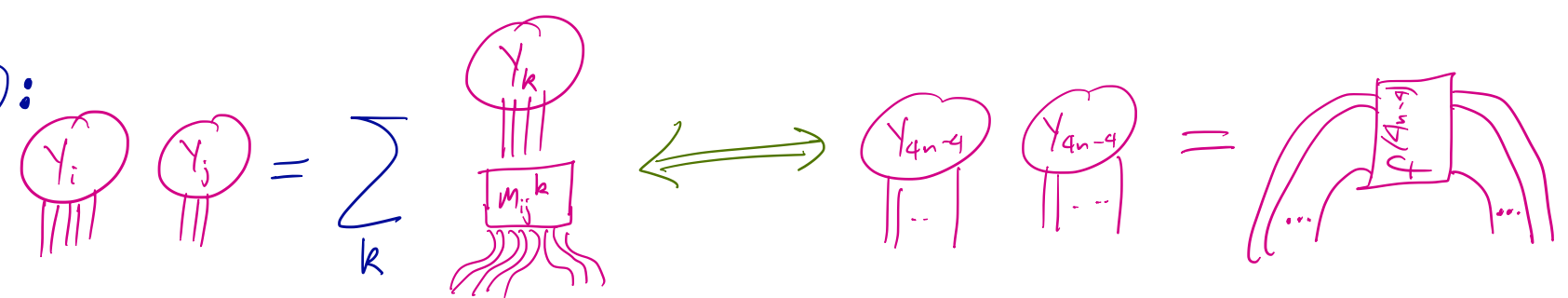
③:



so we can leave out Y_0

This is exactly our skein theoretic presentation for D_{2n} !

④:



Exercise

- calculate the structure coefficients for the algebras
in A_{11} and in A_{29}
- write down the corresponding skein theories for the
free-module categories
- in A_{29} , with $A = f^{(0)} \oplus f^{(1)} \oplus f^{(19)} \oplus f^{(29)}$, derive
formulas in the skein theory for γ_{19} and γ_{29}
in terms of γ_{10} , thereby showing the skein theory
is singly-generated
- observe that this gives Bigelow's skein theory
for E_8 from arXiv:0903.0144
(no one has ever done this exercise!)

Let's return to D_{2n} and compute the minimal idempotents, and fusion rules.

- We have all the idempotents $\begin{array}{c} | \dots | \\ \boxed{f^{(k)}} \\ | \dots | \end{array}$ for $0 \leq k \leq 4n-4$

Are they still minimal, and non-isomorphic?

$$D_{2n}(f^{(k)} \rightarrow f^{(l)}) = \left\{ \begin{array}{c} | \dots | \\ \boxed{f^{(k)}} \\ | \dots | \\ \text{g} \\ | \dots | \\ \boxed{f^{(l)}} \\ | \dots | \end{array} \middle| \begin{array}{l} \text{g a planar diagram} \\ \text{built out of S's, with} \\ k+l \text{ boundary points} \end{array} \right\}$$

Using the evaluation algorithm, we can ensure g has at most one $\begin{array}{c} \text{S} \\ | \dots | \end{array}$.

If $k+l < 4n-4$, that $\begin{array}{c} \text{S} \\ | \dots | \end{array}$ has a cap, and so must be zero!

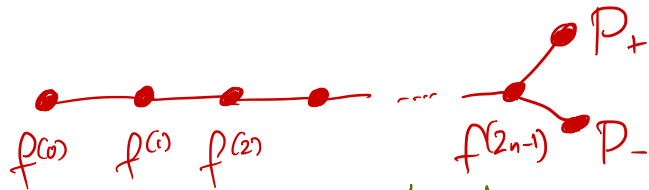
Therefore for $0 \leq k \leq 2n-1$, the $f^{(k)}$ are still minimal idempotents, and pairwise non-isomorphic.

Next:
$$f^{(2n-2)} = \frac{1}{2} \left(f^{(2n-2)} + S \right) + \frac{1}{2} \left(f^{(2n-2)} - S \right) =: P_+ + P_-$$

and the two terms are minimal idempotents which are non-isomorphic. Exercise!

Finally:
$$P_+ \cong f^{(2n-1)}$$
 Exercise: in both directions, the morphism is just S .

Putting this all together, the minimal idempotents are



Exercise: we know how to tensor each of these with a strand.

Explain why this shows every idempotent is a direct sum of these ones.

We can also do these calculations from the perspective of

$$\text{TLJ}(f^{\text{mod}} - (f^{(0)} \oplus f^{(4n-4)}))$$

$$D_{2n}(f^{(k)} \rightarrow f^{(l)}) \equiv \text{TLJ}(f^{(k)} \rightarrow f^{(l)} \oplus (f^{(0)} \oplus f^{(4n-4)}))$$

$$\cong \text{TLJ}(f^{(k)} \rightarrow f^{(l)} \oplus f^{(4n-4-l)})$$

$$\cong \text{TLJ}(f^{(k)} \rightarrow f^{(l)}) \oplus \text{TLJ}(f^{(k)} \rightarrow f^{(4n-4-l)})$$

$$= \begin{cases} \mathbb{C}^2 & \text{if } k=l=2n-2 \end{cases}$$

$$\begin{cases} \mathbb{C}^1 & \text{if } k=l \text{ or } k=4n-4-l, \text{ but } k \neq l \end{cases}$$

$$\mathbb{C}^0 \text{ otherwise}$$

This tells you how $f^{(k)}$ decomposes, but to compute the

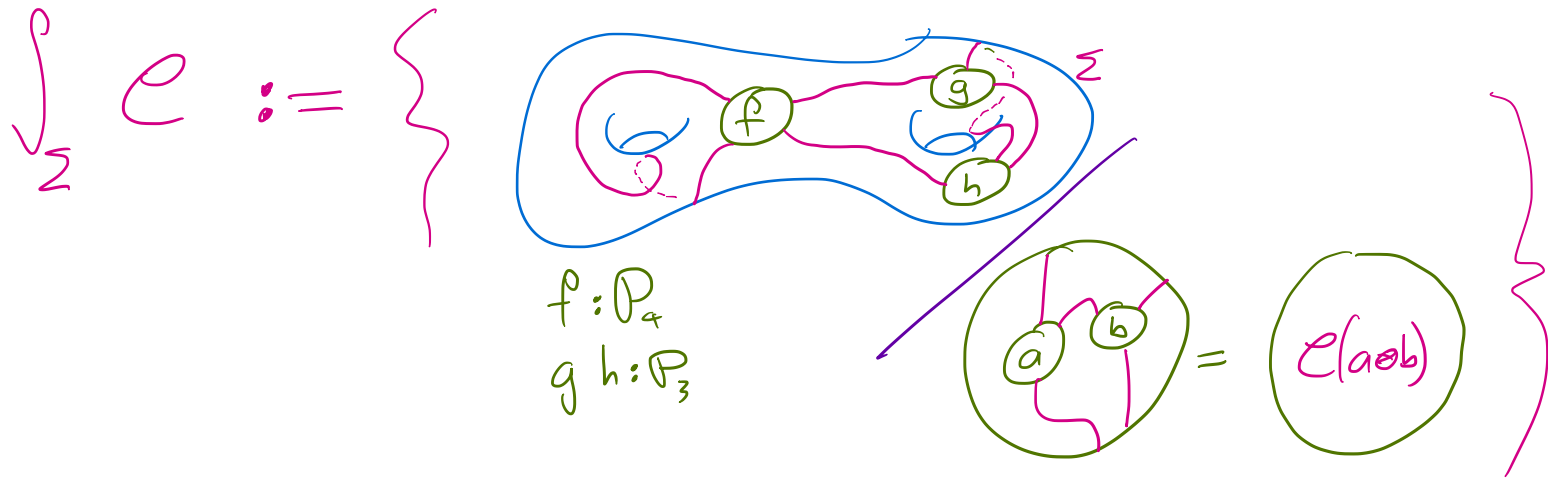
tensor product rules we'd need to use the algebra structure...

Possible next topics

- annular TL, lowest weight decompositions
 - triple point obstructions
- classification of small examples
 - by rank, index, global dimension, ...
- constructing exotic categories using graph planar algebras
- annular categories, Drinfeld centres, tube algebras
- string diagrams for n -categories

Drinfeld centres

Given a planar algebra/pivotal category/skein theory \mathcal{C} ,
we immediately get a vector space



for any oriented surface Σ .

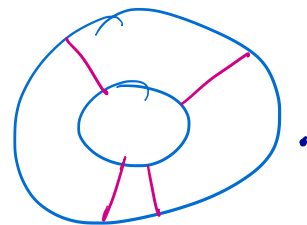
Going down just one dimension, $\int_{S^1} \mathcal{C}$ is a 1-category:

$$\text{Obj } \int_{S^1} \mathcal{C} := \left\{ \text{circle with points } x, y, z \mid x, y, z : \text{Obj } \mathcal{C} \right\}$$

$$\left(\int_{S^1} \mathcal{C} \right) \left(\text{circle with } A, B \rightarrow \text{circle with } x, y \right) := \int_{S^1} \mathcal{C} = \left\{ \text{annulus with paths } g, h, p, q \text{ and points } x, y, A, B \right\} / \sim$$

Composition is just by gluing annuli concentrically.

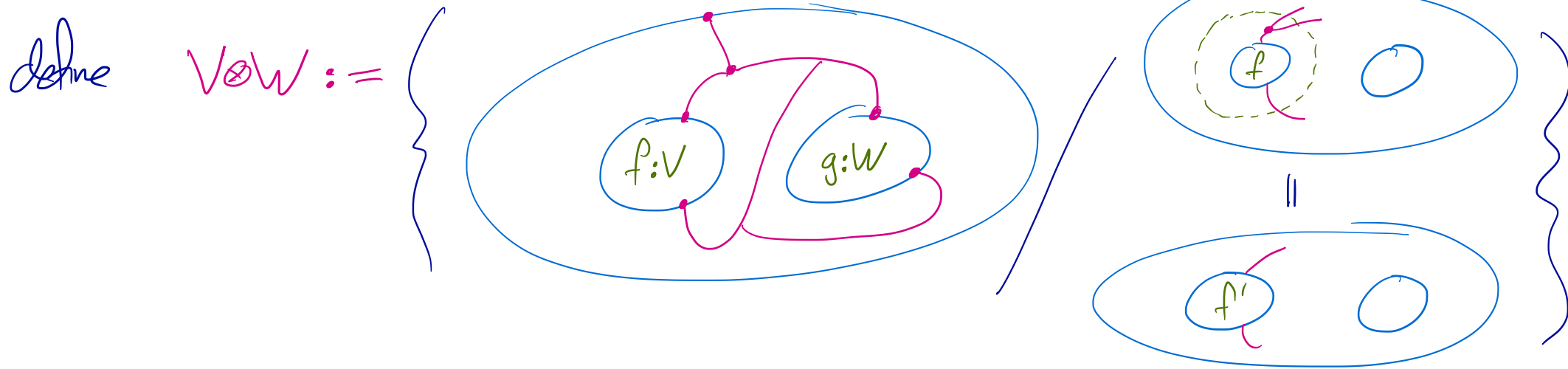
Identities are the morphisms



Exercise explain why checking associativity is the same as checking associativity in $\pi_1(X)$.

While $\int_{S^1} \mathcal{C}$ is only a 1-category, $\text{Rep} \int_{S^1} \mathcal{C}$ naturally has the structure of a braided monoidal category. (Recall $\text{Rep} \mathcal{D} := \text{Fun}(\mathcal{D} \rightarrow \text{Vec})$)

Given two representations $V, W: \text{Rep} \int_{S^1} \mathcal{C}$,



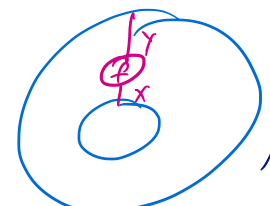
where $f' := V(\text{circle with } f \text{ inside})(f)$

This gives a new representation of the category $\int_{S^1} \mathcal{C}$, by gluing annular diagrams on the outside.

The braiding $V \otimes W \rightarrow W \otimes V$ is given by dragging the pictures around!

Let's understand why (and when) this agrees with the usual
 Drinfeld centre of a monoidal category:

$$\text{Obj } Z(\mathcal{C}) := \left\{ (X: \mathcal{C}, \beta: X \otimes - \xrightarrow{\cong} - \otimes X) \right\}$$

Because there is a functor $\mathcal{C} \rightarrow \int_{S^1} \mathcal{C}$, $\phi_X^Y \mapsto$ 

any $V: \text{Rep} \int_{S^1} \mathcal{C}$ gives a representation of \mathcal{C} (not a module category for \mathcal{C} as a \otimes -category, just a representation of the underlying \mathcal{C} -category)

and if \mathcal{C} is semisimple then every representation is representable, i.e. isomorphic to one of the

form $Y \mapsto \mathcal{C}(X \rightarrow Y)$ for some object $X: \mathcal{C}$.

We'll take that X as the underlying object of the object in $Z(\mathcal{C})$ that we're building.

Now we need $\beta_Y: X \otimes Y \rightarrow Y \otimes X$,

We have

$$\text{id} \in \mathcal{C}(X \rightarrow X) \cong V(\text{circle with } x \text{ on top})$$

and, using pivotality,

$$\begin{aligned} \mathcal{C}(X \otimes Y \rightarrow Y \otimes X) &\cong \mathcal{C}(X \rightarrow Y \otimes X \otimes Y^v) \\ &\cong V(\text{circle with } y, x, y^v \text{ on top}) \end{aligned}$$

$$\cup \left(V(\text{circle with } y, x, y^v \text{ on top}) \right) \left(\text{"id"} \in V(\text{circle with } x \text{ on top}) \right)$$

- Exercises
- check β_Y is an isomorphism
 - check β_Y is natural in Y
 - check β_Y is monoidal in Y

Exercise extend this to a functor $\text{Rep}_{\text{fs}} \mathcal{C} \rightarrow Z(\mathcal{C})$

(hint: use Yoneda. bonus: is this functor co- or contra-variant?)

In fact, this functor was 'the hard direction' — we needed to use semisimplicity of \mathcal{C} to construct it.

The 'easy direction' $Z(\mathcal{C}) \rightarrow \text{Rep}_{\text{fs}} \mathcal{C}$ should send an object with a half-braiding (X, β) to the representation

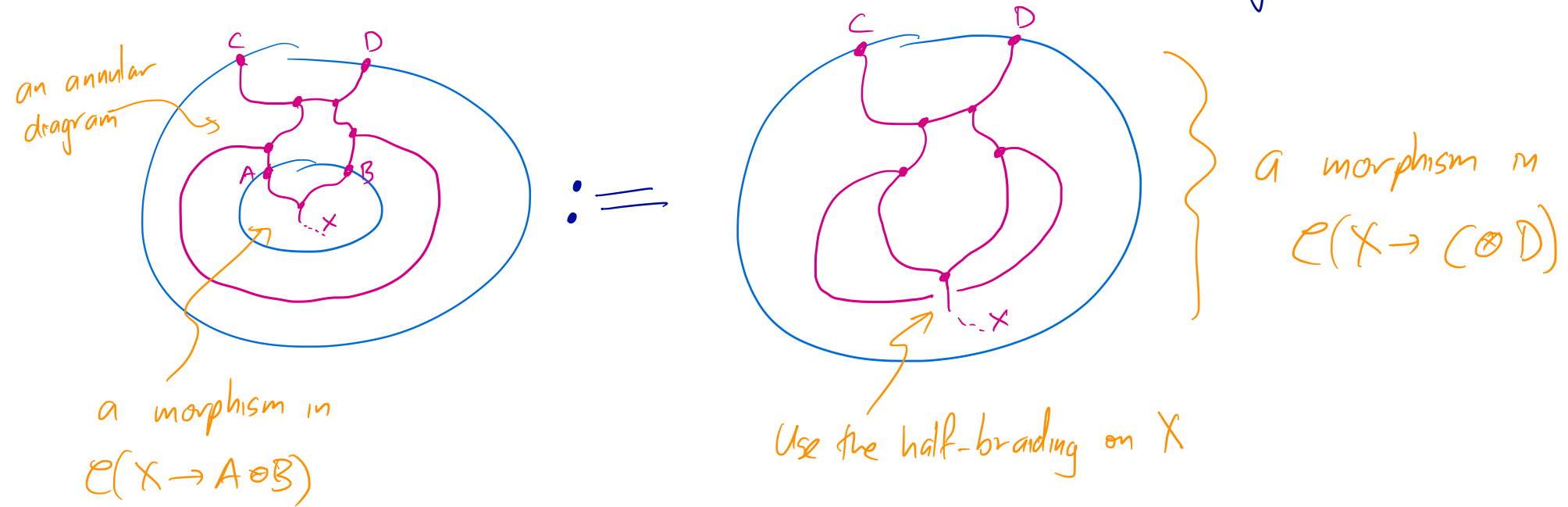
$$V(\text{circle with points } A, B, C) := \mathcal{C}(X \rightarrow A \otimes B \otimes C)$$

but this isn't well-defined! We only have a cyclic ordering on the points around the circle, but need a linear ordering to take homs to.

Of course, the half-braiding saves us, giving isomorphisms

$$\mathcal{C}(X \rightarrow A \otimes B \otimes C) \rightarrow \mathcal{C}(X \otimes C^{\vee} \rightarrow A \otimes B) \rightarrow \mathcal{C}(C^{\vee} \otimes X \rightarrow A \otimes B) \rightarrow \mathcal{C}(X \rightarrow C \otimes A \otimes B)$$

We also need to define the action of annular diagrams:



We still have many proof obligations:

- the functors $Z(\mathcal{C}) \rightleftarrows \text{Rep} \int_{S^1} \mathcal{C}$ form an equivalence
- at least one of the functors is monoidal (Exercise: why just one?)
- that functor is braided
- (we may also want to define and compare pivotal structures on either side)

Let's check monoidality of the functor $\mathcal{V}: \mathcal{Z}(C) \rightarrow \text{Rep}_{S'} C$.

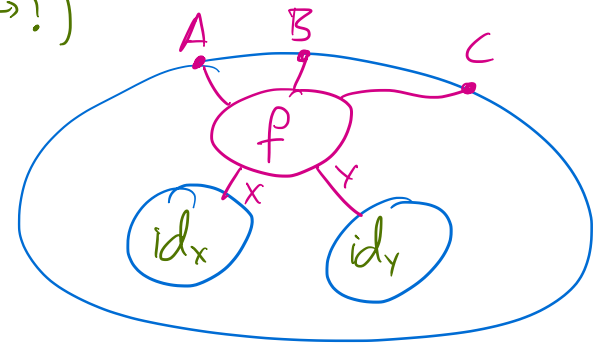
$$\mathcal{V}((X, \beta) \otimes (Y, \gamma)) \left(\text{circle with points } A, B, C \right) \equiv C(X \otimes Y \rightarrow A \otimes B \otimes C)$$

$$= (X \otimes Y, \begin{matrix} X \otimes Y \otimes Z \xrightarrow{\beta} X \otimes Z \otimes Y \\ \xrightarrow{\gamma} Z \otimes X \otimes Y \end{matrix})$$

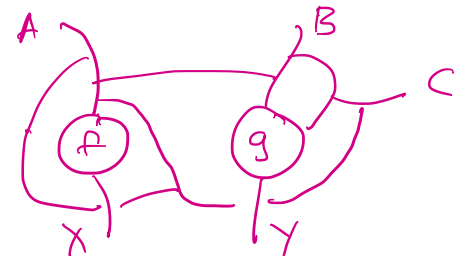
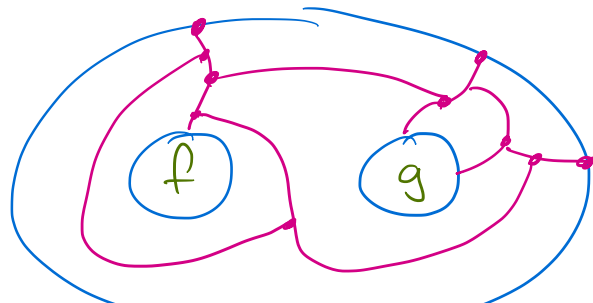
$$(\mathcal{V}(X, \beta) \otimes \mathcal{V}(Y, \gamma)) \left(\text{circle with points } A, B, C \right) \equiv \left\{ \text{Diagram with } f, g \text{ and paths} \right\} / \sim$$

$f: C(X \rightarrow ?)$
 $g: C(Y \rightarrow ?)$

We can map one way via



and the other via



This interpretation of $Z(e)$ as $\text{Rep}_S \mathcal{E}$ explains why physicists think of $Z(e)$ as classifying the 'point defects' or 'pointlike excitations' of the 2d lattice model realising \mathcal{E} .

An object of $Z(e)$ is a collection of vector spaces describing the possible quantum states of an excitation, along with rules (the annular action) for how these states interact with the surrounding state of the bulk theory.

Let's describe the simplest example of a Drinfeld centre.

(to physicists, this example is "the toric code")

Let $\mathcal{C} = \text{Rep } \mathbb{Z}/2\mathbb{Z} = \text{TLJ}_{S=1} = \left\langle \begin{array}{l} \text{no} \\ \text{generators} \end{array} \middle| 0=1, \sum = 0 \right\rangle$.

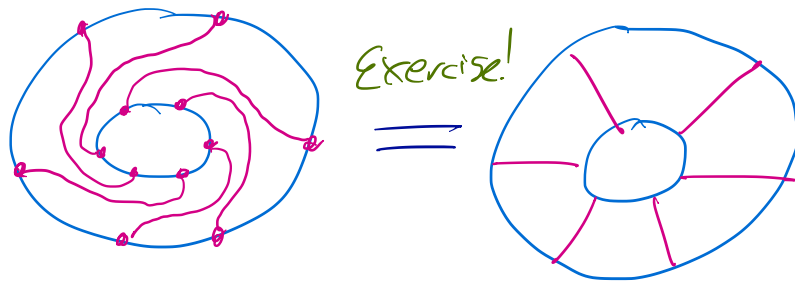
A representation V of $\int_S \mathcal{C}$ consists of:

- for each $n \in \mathbb{N}$, a vector space $V_n = V(\text{circle with } n \text{ boundary points})$

- however, $V(\text{torus with } n \text{ radial lines})$ gives an isomorphism (Exercise: what is the inverse?)

$$V_{n+2} \cong V_n$$

and since



Exercise!

wherever we put the cap we get the same isomorphism,

and so the representation V is entirely determined by V_0 and V_1 , and the annular tangles between these.

In fact, there are no maps between V_0 and V_1 , so we get a direct sum decomposition into two pieces.

(equivalently, in any irrep $V_0=0$ or $V_1=0$)

What is $\left(\int_S e\right)(\bigcirc \rightarrow \bigcirc)$? Just $\mathbb{C}\{\bigcirc, \bigcirc\} / \{\bigcirc = \bigcirc\}$

Similarly $\left(\int_S e\right)(\dot{\bigcirc} \rightarrow \dot{\bigcirc}) = \mathbb{C}\{\bigcirc, \bigcirc\} / \{\bigcirc = \bigcirc\}$

From these facts we have the classification of reps:

$$V^{\text{triv}} = \left\{ \text{disk with 4 red dots and paths} \right\} / \left\{ \text{annulus} = + \text{disk} \right\}$$

"magnetic" \downarrow

$$V^m = \left\{ \text{disk with 4 red dots and paths} \right\} / \left\{ \text{annulus} = - \text{disk} \right\}$$

"electric" \downarrow

$$V^{e_{\frac{1}{2}}} = \left\{ \text{disk with 4 red dots and paths} \right\} / \left\{ \text{annulus} = + \text{disk} \right\}$$

$$V^{em} = \left\{ \text{disk with 4 red dots and paths} \right\} / \left\{ \text{annulus} = - \text{disk} \right\}$$

Exercises

- calculate the tensor products
- calculate the braiding maps
- compute the S & T matrices
- verify $\mathbb{Z}(\text{Rep } \mathbb{Z}/2\mathbb{Z})$ is modular.

From this perspective we can describe the functor

$$G: \mathcal{C} \otimes \mathcal{C}^{\text{loop}} \longrightarrow Z(\mathcal{C}) = \text{Rep} \int_{S^1} \mathcal{C}$$

whenever \mathcal{C} is braided.

Define $G(X, Y) = \left\{ \begin{array}{c} \text{Diagram 1: A genus-1 surface with a central hole. A pink path starts at point X on the inner boundary, goes right to point Y, then loops around the hole, and returns to X. Additional pink paths are shown: one from X to point A on the outer boundary, one from Y to point B, and one from Y to point C. Points A, B, and C are marked on the outer boundary. The path from Y to C goes down and then right. The path from Y to B goes up and then right. The path from X to A goes up and then left. } \end{array} \right\} / \sim$

= $\begin{array}{c} \text{Diagram 2: A genus-1 surface with a central hole. A pink path starts at point X on the inner boundary, goes right to point Y, then loops around the hole, and returns to X. A point Z is marked on the top of the inner boundary. } \end{array} = \begin{array}{c} \text{Diagram 3: A genus-1 surface with a central hole. A pink path starts at point X on the inner boundary, goes right to point Y, then loops around the hole, and returns to X. } \end{array}$

Observe this is monoidal:

$$G(X, Y) \otimes G(W, Z) = \left\{ \begin{array}{c} \text{Diagram 4: Two genus-1 surfaces side-by-side. The left one has points X and Y on its inner boundary. The right one has points W and Z on its inner boundary. } \end{array} \right\} = \begin{array}{c} \text{Diagram 5: A genus-2 surface with two holes. Pink paths connect X to Y and W to Z, each looping around its respective hole. } \end{array} = \begin{array}{c} \text{Diagram 6: A genus-2 surface with two holes. Pink paths connect X to W and Y to Z, each looping around its respective hole. } \end{array} \Bigg\}$$

Exercise check the details, and confirm the braiding is preserved.

\Downarrow
 $G(X \otimes W, W \otimes Z)$

Exercises

- When \mathcal{C} is symmetric monoidal, show this functor factors through

$$\mathcal{C} \otimes \mathcal{C}^{\text{loop}} \xrightarrow{\otimes} \mathcal{C} \rightarrow \mathbb{Z}(\mathcal{C})$$

- When \mathcal{C} is modular, show this functor is an equivalence.

How do we concretely calculate $Z(\mathcal{C})$?

When \mathcal{C} is finitely semisimple, $\int_{S^1} \mathcal{C}$ is equivalent (after Cauchy completion) to a certain finite-dimensional associative algebra, the tube algebra.

$$\text{Tube}(\mathcal{C}) := \bigoplus_{X, Y, Z: \text{lrr} \mathcal{C}} \mathcal{C}(X \otimes Z \rightarrow Z \otimes Y) / \text{for all } f: \mathcal{C}(Z \rightarrow Z')$$

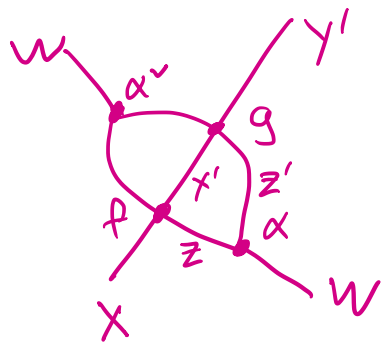
$$g: \mathcal{C}(X \otimes Z' \rightarrow Z \otimes Y)$$

Composition sends $f: \mathcal{C}(X \otimes Z \rightarrow Z \otimes Y)$

$$g: \mathcal{C}(X' \otimes Z' \rightarrow Z' \otimes Y')$$

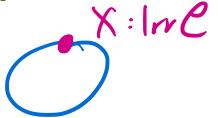
to

$$\delta_{X'=Y} \sum_{W: \text{lrr} \mathcal{C}} \sum_{\substack{\alpha: \text{a basis} \\ \text{of} \\ \mathcal{C}(W \rightarrow ZZ')}} \alpha^\vee$$

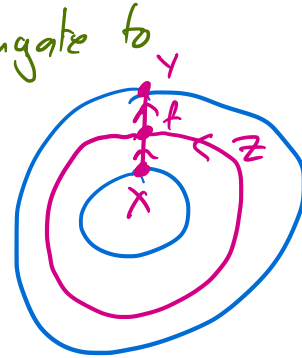


(Here α^\vee is the dual basis element under the pairing $\mathcal{C}(W \rightarrow ZZ') \otimes \mathcal{C}(ZZ' \rightarrow W) \xrightarrow{\circ} \mathcal{C}(W \rightarrow W) \cong \mathbb{C}$)

Exercise

- every object in $\int_{S^1} \mathcal{C}$ is a direct sum of objects 

- every morphism in $\int_{S^1} \mathcal{C}$ is conjugate to a matrix of morphisms



$$x, y, z: \text{lnc}$$
$$f: \mathcal{C}(xz \rightarrow zy)$$

- use this to prove the equivalence

$$\text{Mat}(\text{Kor}(\text{Tube } \mathcal{C})) \cong \text{Mat}(\text{Kor}(\int_{S^1} \mathcal{C}))$$

(hint: to define a functor out of $\text{Mat}(\text{Kor}(\mathbb{D}))$ it suffices to define a functor out of \mathbb{D} .)

Let's calculate $\text{Tube}(\text{Rep } \mathbb{Z}/2\mathbb{Z})$.

$$\mathbb{C}(X \rightarrow Z \rightarrow Y) \cong \begin{cases} \mathbb{C} & \text{if } X=Y \\ 0 & \text{otherwise} \end{cases}$$

So $\text{Tube}(\text{Rep } \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{C}^4$. But what is the algebra structure?

We could compute it using the formula!

$$fg = \delta_{X=Y} \sum_{w: \text{line}} \sum_{\substack{\alpha: \text{a basis} \\ \text{of } \mathbb{C}(w \rightarrow \mathbb{Z}\mathbb{Z}')}} \text{Diagram}$$

But there's a shortcut: the S factor shows the algebra has a direct sum decomposition, into $X=Y = \mathbb{C}^{\text{triv}}$ and $X=Y = \mathbb{C}^{\text{sign}}$ pieces.

Since $\text{Rep } \mathbb{Z}/2\mathbb{Z}$ is unitary, $\text{Tube}(\text{Rep } \mathbb{Z}/2\mathbb{Z})$ must be semisimple,

$$\text{So } \text{Tube}(\text{Rep } \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}.$$

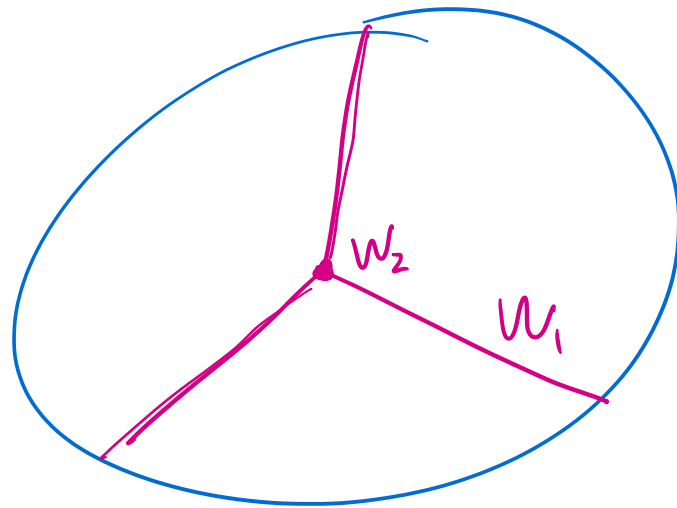
Exercise compute $\text{Tube}(\text{Rep } \mathbb{Z}/2\mathbb{Z})$ and $\text{Tube}(\text{Fib})$ explicitly.

What is ... a string diagram n -category?

A stratification of an n -manifold W is a sequence

$$W = W_0 \supseteq W_1 \supseteq \dots \supseteq W_n$$

so $W_k \setminus W_{k+1}$ is a codimension k submanifold of W .

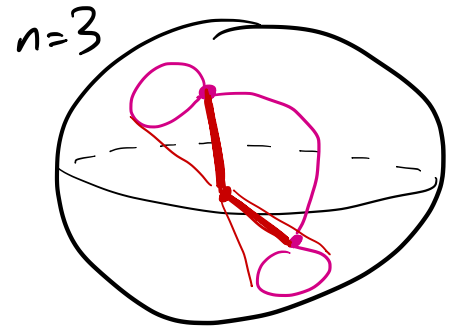
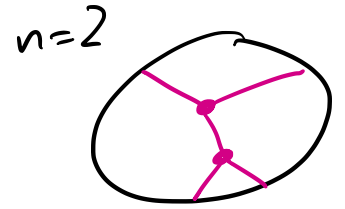


It is a string diagram stratification if

every $x \in W_k \setminus W_{k+1}$ has a nbhd in W

which is a product of its nbhd in W_k

and the cone over a string diagram stratification of some $(n-k-1)$ -sphere.

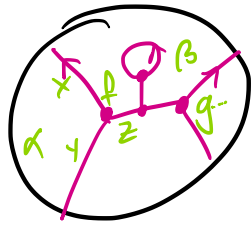


Equivalently, if:

- it is the 0-ball with its unique stratification
- it is $W \times I$ for some W with an SD stratification
- it is $\text{cone}(W)$, where W is an SD stratification of a sphere
- it is $W_1 \sqcup W_2$, where W_1 and W_2 have SD stratifications, or
- it is W glued to itself along Y , where W has an SD stratification, and both restrictions to Y give the same stratification.

[Siebenmann '72!]

If " \mathcal{C} " is an n -category, an m -dimensional \mathcal{C} -string diagram is an SD-stratified ball W^m with each $(m-k)$ ball in $W_k \setminus W_{k+1}$, labelled by some k -morphism of \mathcal{C} .



Any reasonable definition of a (pivotful) n -category lets you evaluate a \mathcal{C} -string diagram to an m -morphism, so isotopic \mathcal{C} -string diagrams have the same evaluation.

$$\text{ev} \left(\text{diagram} \right) = \text{diagram}$$

We will define a n -category by axiomatising just this observation.

An n -dimensional signature \mathcal{L} consists of

* oriented, unoriented,
spin, conformal,
equipped with a map to \mathbb{T} ,
etc

— for each $0 \leq k \leq n$

— for each oriented* k -ball X ,

— for each string diagram stratification α of ∂X

— for each labelling c of each stratum $s \in \alpha$ by an element of \mathcal{L}^{k-1} (normal disc of s)

a set $\mathcal{L}^k(X; c)$.

Example: $\mathcal{L}^0(\cdot) = \{\cdot\}$, $\mathcal{L}^1(\curvearrowright) = \{\curvearrowright, \curvearrowleft\}$,

$\mathcal{L}^2(\text{circle with 4 dots}) = \{\text{circle with } \times\}$, $\mathcal{L}^2(\text{circle with 4 dots}) = \{\text{circle with } \cup, \text{circle with } \cap\}$

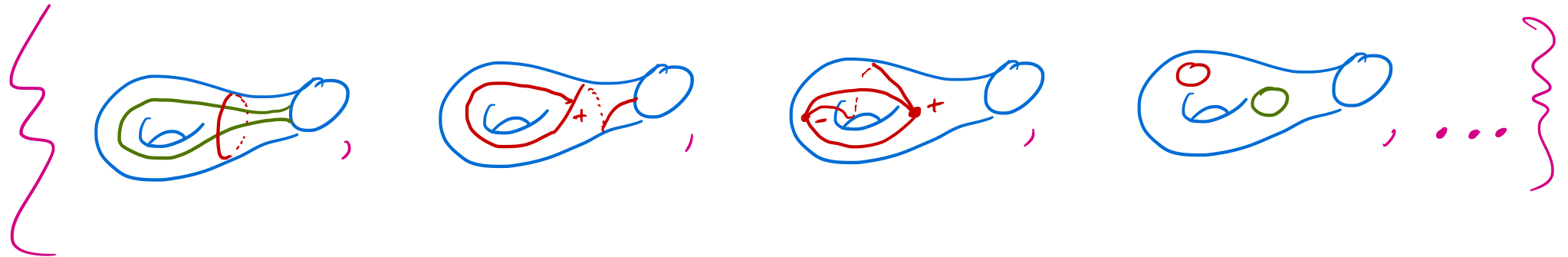
all other \mathcal{L}^2 empty.

For a fixed signature \mathcal{I} , a string diagram on M^n is a string diagram stratification of M , with each stratum labelled by an element of \mathcal{I} (normal disc)

Example with \mathcal{I} as before,

\mathcal{I} string diagrams on 

||



A string diagram n-category \mathcal{C} consists of:

- a n -signature \mathcal{L}
- for each n -ball X with a \mathcal{L} -string diagram c in ∂X

a subspace

$$\mathcal{U}(X; c) \subset \mathcal{C} \left\{ \begin{array}{l} \mathcal{L}\text{-string diagrams on } X \\ \text{with boundary } c \end{array} \right\}$$

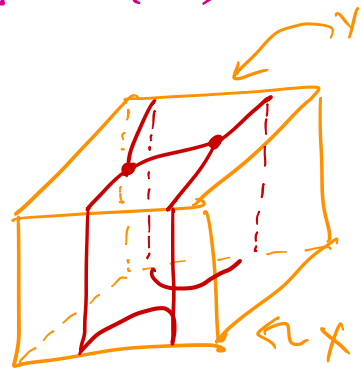
- such that \mathcal{U} forms an ideal with respect to gluing balls
- if f and g are \mathcal{L} -string diagrams on X with the same boundary c , such that f and g are isotopic rel ∂ , then $f - g \in \mathcal{U}(X; c)$.

Without pinning ourselves down to a particular 'traditional' definition of (weak!) n -category, let's sketch the construction of one.

$$\hat{\mathcal{C}}^0 = \mathcal{I}^0(\cdot), \quad \hat{\mathcal{C}}^1(X \rightarrow Y) = \left\{ \begin{array}{l} \text{string diagrams on } [0,1] \text{ with} \\ X \text{ at } 0 \text{ and } Y \text{ at } 1 \end{array} \right\}$$

$$\hat{\mathcal{C}}^{k < n}(X \rightarrow Y) = \left\{ \begin{array}{l} \text{string diagrams on } [0,1]^k \text{ with} \\ X \text{ on } [0,1]^{k-1} \times \{0\}, \quad Y \text{ on } [0,1]^{k-1} \times \{1\} \\ (\partial X = \partial Y) \times I \text{ on } \partial([0,1]^{k-1}) \times I \end{array} \right\}$$

$$\hat{\mathcal{C}}^n(X \rightarrow Y) = \mathcal{C} \left\{ \text{string diagrams as above} \right\} / \mathcal{U}([0,1]^k; X, Y)$$



Conjecture/hypothesis

Every pivotal weak n -category is equivalent to a string diagram n -category.

Payoffs:

- straightforward definitions of 'sphere modules'
- Hochschild cohomology for n -categories
- higher dimensional analogues of idempotent completion/algebra completion
- construction of a 4-category from Khovanov homology
- a proof of stabilisation
(k -boring n -categories = $(k+1)$ -boring n -categories when $k \geq \frac{n}{2} + 1$)
using transversality

} (arXiv:1009.5025)

(coming soon, see also arXiv:1405.09566)

(coming soon)

(someone should do it!)

How do we construct potential new tensor categories?

Suppose we suspect the existence of some tensor category \mathcal{C} with a particular fusion ring $K_0(\mathcal{C})$ in particular suppose we know the principal graph for some object, e.g.



Any monoidal category \mathcal{C} embeds via a monoidal functor into the endofunctors of the underlying 1-category $\mathcal{M} = \mathcal{C}$.

$$\begin{array}{l} \mathcal{G}: \mathcal{C} \xrightarrow{\otimes} \text{End}(\mathcal{M}) \\ X \longmapsto X \otimes - \end{array}$$

If \mathcal{C} is semisimple, with n simple objects, then $\mathcal{M} \cong \text{Vec}^{\oplus n}$,

and $\text{End}(\mathcal{M})$ is semisimple, with simple object E_{ij} , defined

by the rule
$$E_{ij}(\mathbb{C}_k) = \begin{cases} \mathbb{C}_i & \text{if } j=k \\ 0 & \text{otherwise} \end{cases}$$

 \hookrightarrow the additive generator of the k -th summand of $\text{Vec}^{\oplus n}$

A general object of $\text{End}(\mathcal{M})$ is (up to iso) a direct sum of these, i.e. a matrix in $M_n(\mathbb{N})$, which we can interpret as the adjacency matrix of a directed graph.

Now
$$g(X:e)(\mathbb{C}_k) = X \otimes X_k = \bigoplus n_k^e X_e$$

 \hookrightarrow the k -th simple of e

where n_k^e is the adjacency matrix for the principal graph, so

$$g(X) = \Gamma$$

Next we would like to calculate

$$\begin{aligned}
 \text{End}(\mathcal{M})(\textcircled{H} \rightarrow \underline{\Lambda}) &= \bigoplus_{k \in \mathbb{K}, s \in n} \text{Vec}(\textcircled{H}(\mathbb{C}_k) \rightarrow \underline{\Lambda}(\mathbb{C}_k)) \\
 &\stackrel{\substack{\uparrow \\ \text{graphs, interpreted as} \\ \text{functors}}}{=} \bigoplus_k \text{Vec}(\bigoplus_k \Theta_k^l \mathbb{C}_l \rightarrow \bigoplus_k \Lambda_k^l \mathbb{C}_l) \\
 &= \bigoplus_{k, l} \Theta_k^l \Lambda_k^l
 \end{aligned}$$

This has as a basis pairs of edges, one in \textcircled{H} , one in $\underline{\Lambda}$ with the same source and target.

Composition of basis elements is easy:

$$(e \in \textcircled{H}, f \in \underline{\Lambda}) \gg (g \in \underline{\Lambda}, h \in \Delta) = \begin{cases} (e, h) & \text{if } f=g \\ \text{zero} & \text{otherwise} \end{cases}$$

How do we take tensor products of graphs?

$\textcircled{H} \otimes \textcircled{\Lambda} =$ length two paths in K_n , first edge in \textcircled{H} , second edge in $\textcircled{\Lambda}$

$\textcircled{H}^{\otimes n} =$ length n paths in \textcircled{H}

How do we tensor morphisms?

$(e \in \textcircled{H}, f \in \textcircled{\Lambda}) \otimes (g \in \textcircled{\Delta}, h \in \textcircled{\Omega})$

$= \left(e \gg g \in \textcircled{H} \otimes \textcircled{\Delta}, f \gg h \in \textcircled{\Omega} \otimes \textcircled{\Omega} \right)$

if (e, g) and (f, h) are composable pairs
Zero otherwise

Now: $\text{End}(\mathcal{M})(\Gamma^{\otimes n} \rightarrow \Gamma^{\otimes m}) =$

$\left\{ \sum \text{pairs of paths } (\gamma, \gamma'), \text{ of length } n \text{ and } m, \right.$
 $\left. \text{with the same source and target} \right\}$

Composition is just $(\gamma, \gamma') \circ (\gamma'', \gamma''') = \delta_{\gamma', \gamma''} (\gamma, \gamma''')$

Tensor is concatenation of paths, if possible.

At this point we know there must exist an embedding

$$\langle X: \mathcal{C} \rangle \longrightarrow \langle \Gamma: \text{End}(\mathcal{M}) \rangle$$

with a completely combinatorial description of the target monoidal category!

In fact, it gets better. $\text{End}(\mathcal{M})$ is a pivotal category,
 with duality given by taking right adjoints

(which exist as \mathcal{M} is semisimple, and in fact are biadjoint)

which at the level of the graphs is orientation reversal.

In fact $\Gamma^{vv} = \Gamma$, so we could take the pivotal isomorphism

$$\Sigma : \mathbb{1}_{\text{End}(\mathcal{M})} \rightarrow vv$$

to be the identity.

This is a bad idea!

If \mathcal{C} is pivotal, each simple X_i has a categorical dimension

$$\dim(X) := \psi \circ \text{tr} =$$

$$\in \mathbb{C}$$

Theorem If we give $\text{End}(\mathcal{M})$ the pivotal structure

$$\tau_{E_{ij}} = \frac{\dim(X_i)}{\dim(X_j)} \mathbb{1}_{E_{ij}}$$

then $\mathcal{C} \rightarrow \text{End}(\mathcal{M})$ is a pivotal embedding.

Exercise verify this (replacing the ratio with its inverse if needed!!)

(c.f. arXiv:1007.3173, arXiv:1810.06076, arXiv:1808.00323)

Idea: ① From the principal graph Γ , deduce the existence of an element $S: \mathcal{C}(1 \rightarrow X^{\otimes n})$

satisfying certain planar equations $\{r_i\}$

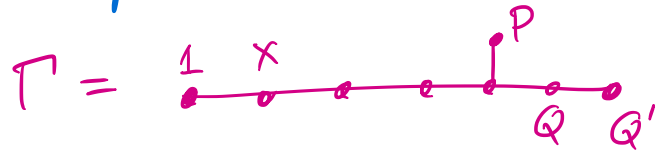
② solve these equations in $\text{End}(\mathcal{M})(1 \rightarrow \Gamma^{\otimes n})$

③ this gives an embedding $F: \langle S | r_i \rangle \hookrightarrow \text{End}(\mathcal{M})$

④ find more equations if necessary, perhaps by observing equations satisfied by the solution in $\text{End}(\mathcal{M})$

⑤ use these equations to show $\langle S | r_i \rangle$ is evaluable and has the desired simple objects and fusion rules.

Let's try to implement this for



We see $\dim \mathcal{C}(1 \rightarrow X^{\otimes n}) = \#$ of loops of length n based at 1

n	0	2	4	6	8	10
$\dim \mathcal{C}$	1	1	2	5	14	43
$\dim TL$	1	1	2	5	14	42

Therefore $\mathcal{C}(1 \rightarrow X^{\otimes n}) = TL(1 \rightarrow X^{\otimes n})$ for $n \leq 8$

$$\mathcal{C}(1 \rightarrow X^{\otimes 10}) = TL(1 \rightarrow X^{\otimes 10}) \oplus \mathbb{C} \left\{ \begin{array}{c} \text{||||} \\ \text{S} \\ \text{||||} \end{array} \right\}$$

We can pick this S so $\begin{array}{c} \text{||||} \\ \text{S} \\ \text{||||} \end{array} = \text{zero}$

and $\begin{array}{c} \text{||...||} \\ \text{S} \\ \text{||...||} \end{array} = \omega \begin{array}{c} \text{||...||} \\ \text{S} \\ \text{||...||} \end{array}$ for some 10-th root of unity ω .

In $\mathcal{E}(X^{\otimes 5} \rightarrow X^{\otimes 5})$ we see $\boxed{\begin{array}{c} | | | | \\ p(5) \\ | | | | \end{array}}$ must break up
 as a sum of two non-isomorphic projections

$$\boxed{\begin{array}{c} | | | | \\ p(5) \\ | | | | \end{array}} = \left(\alpha \boxed{\begin{array}{c} | | | | \\ p(5) \\ | | | | \end{array}} + S \right) + \left((1-\alpha) \boxed{\begin{array}{c} | | | | \\ p(5) \\ | | | | \end{array}} - S \right)$$

(we use our remaining freedom to renormalize S here)

In fact, since $\dim P = \text{FP dim } P = P\text{-entry of FP-eigenvector of } \Gamma$
assume unitarity!

$$= \frac{1}{4} (1 + \sqrt{5}) \sqrt{7 + \sqrt{5} + \sqrt{6(5 + \sqrt{5})}} \sim 3.21834$$

so we know α .

$$\left(\alpha \boxed{\begin{array}{c} | | | | \\ p \\ | | | | \end{array}} + S \right)^2 = \left(\alpha \boxed{\begin{array}{c} | | | | \\ p \\ | | | | \end{array}} + S \right) \text{ gives us a quadratic in } S$$

and we're done!

Exercise implement 'the idea'

- Show there are finitely many solutions to these equations in $\text{End}(\text{Vec}^{\oplus 8})(1 \rightarrow E_8^{\otimes 10})$

- Show these equations are enough to evaluate and identify all the idempotents

- Check the principal graph really is E_8

- Conclude there exists a fusion category with principal graph E_8

- In fact, classify such categories using your solutions.

See [arXiv:0909.4099](https://arxiv.org/abs/0909.4099) for the construction of "the most exotic known fusion categories".