Q1 $\mathrm{Tr}(S^{2n}) = 0$ if $\mathrm{ord}(S^2) \nmid n$

Q2 $\mathrm{Tr}(S^n)$ is an invariant of $\mathrm{Rep}(H)$ as tensor category for $n \in \mathbb{N}$.

§1 Order of the antipode and semisimplicity of Hopf algebras

Maschke's thm for finite groups

Let G be a finite group. Then $\Bbbk[G]$ is semisimple iff $|G| \neq 0$ in $\Bbbk$.

NOTE: $\Lambda = \sum_{g \in G} g \in \Bbbk[G]$ is a left (& right) integral. Then, $\varepsilon(\Lambda) = \sum_{g \in G} \varepsilon(g) = \sum_{g \in G} 1$

$= |G|$

Thus, Maschke's thm says
$\Bbbk[G]$ is s.s. $\iff$ $\varepsilon(\Lambda) \neq 0$ in G

Maschke's thm for Hopf algebras

Let H be a f-d. Hopf algebra and Λ be a left integral and ε, the counit.
Then H is s.s. $\iff$ $\varepsilon(\Lambda) \neq 0$.

Radford's trace formula

Let λ be a right integral of H^* & $\Lambda \neq 0$ a left integral of H.
then $\lambda(\Lambda) \neq 0$
We can assume $\lambda(\Lambda) = 1$
Then $\mathrm{Tr}(S^2) = \lambda(1)\,\varepsilon(\Lambda)$

$\therefore$ If $Tr(S^2) \neq 0 \iff \lambda(1) \neq 0$ & $\varepsilon(\Lambda) \neq 0$

$\iff H$ and H^* are semisimple.

(This can be proved in a categorical way).

<u>Corollary</u>: $Tr(S^2) \neq 0 \iff H$ & H^* are semisimple

$\iff S^2 = Id$ (char $\mathbb{k} = 0$, Larson-Radford)

$\implies S^2 = Id$ (char $\mathbb{k} > 0$ Etingof-Gelaki)

<u>Exercise</u>: Conversely, statement in char $\mathbb{k} > 0$ is false.

<u>Corollary</u>: $Tr(S_H^2) = 0$ (where H or H^* is not s.s.) OR $\dim(H)$ (when H, H^* are s.s.)

- $S^2 = Id_H$ has consequences.

$\Downarrow$ $j: V \to V^{**}$

$v \mapsto \hat{v}$ where $\hat{v}(f) = f(v)$

j is an H-module map

So, j is a ~~spherical~~ pivotal structure.

(This pivotal structure is canonical, unique)

<u>Exercise</u>: $j: V \to V^{**}$ is a morphism in $Rep(H)$ provided $S^2 = Id$

§2 FROBENIUS - SCHUR INDICATOR

For this section, assume $\mathbb{k} = \mathbb{C}$

1) Let H be a s.s. Hopf algebra. The distinguished grouplike element of H^* is trivial (i.e. equal to ε).

Pf: By Maschke's thm, $\varepsilon(\Lambda) \neq 0$
Assume $\varepsilon(\Lambda) = 1$ (after rescaling Λ)

then $\Lambda a = \alpha(a)\Lambda$ where $\alpha \in G(H^*)$

$$\Rightarrow \varepsilon(\Lambda)\varepsilon(a) = \alpha(a)\varepsilon(\Lambda)$$
$$\Rightarrow \varepsilon(a) = \alpha(a)$$
$$\Rightarrow \varepsilon = \alpha$$

Similarly, the distinguished grouplike element of H is also trivial. (since it is also s.s.)

Exercise: The nonzero left and right integral of any Taft algebra are linearly independent. ($\therefore$ left & right integral are not the same in general)

Defn: (Linchenko - Montgomery)
Let H be a f.d, s.s. Hopf alg. over $\mathbb{C}$ ($\mathbb{k}$) and Λ a normalized integral, i.e. $\varepsilon(\Lambda) = 1$

For $V \in \operatorname{Rep}(H)$
$$\nu_n(V) = \chi_V(\Lambda^{[n]})$$
where χ_V is the character afforded by Λ and

$\Lambda^{[n]}$

diagonalize Λ until we get n copies

$$\Lambda^{[n]} := m_H \left(\sum \Lambda_{(1)} \otimes \Lambda_{(2)} \otimes \cdots \otimes \Lambda_{(n)} \right)$$

<u>Example</u>: $H = \mathbb{C}[G]$

$$\Lambda = \frac{1}{|G|} \sum_{g \in G} g$$

diagonalize Λ

Recall $\Delta g = g \otimes g$

So, $$\Lambda^{[n]} = \frac{1}{|G|} m_H \Big(\sum_{g \in G} \underbrace{g \otimes g \otimes \cdots \otimes g}_{n \text{ terms}} \Big)$$

$$= \frac{1}{|G|} \sum_{g \in G} g^n$$

If $V \in \mathrm{Rep}(G)$

$$\nu_n(V) = \chi_V(\Lambda^{[n]}) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g^n)$$

Define $\Theta_n(g) = \# \{ x \in G \mid x^n = g \}$

Notice that Θ_n is a class function

Irr. characters form basis of class fns.

<u>CLAIM</u>: $$\Theta_n = \sum_{V \in \mathrm{Irr}(G)} \nu_n(\chi_V)\, \chi_V$$

- $$\Theta_n(1) = \sum_{V \in \mathrm{Irr}(G)} \nu_n(\chi_V)\, \chi_V(1)$$

$$= \sum_{V \in \mathrm{Irr}(G)} \nu_n(\chi_V)\, \dim(V)$$

But, $\sum_{V \in \mathrm{Irr}(G)} \chi_V \dim V = \chi_{\mathbb{C}[G]} = |G| \delta_1$

$$= \begin{cases} 0 & g \neq 1 \\ |G| & g = 1 \end{cases}$$

So, we want to compute

$$\nu_n(\mathbb{C}[G]) = \frac{1}{|G|} \sum_{g \in G} \chi_{\mathbb{C}[G]}(g^n)$$

$$= \sum_{g \in G} \delta_1(g^n)$$

$$= \#\{g \in G \mid g^n = 1\}$$

$\nu_2(\mathbb{C}[G]) =$ # involutions in G

Frobenius - Schur Theorem for Hopf algebras:

(Linchenko - Montgomery)

① If $V \in \mathrm{Rep}(H)$ is irreducible

$$\nu_2(V) = \begin{cases} \pm 1 & \text{if } V \cong V^* \\ 0 & \text{if } V \not\cong V^* \end{cases}$$

1 : if V admits H-invariant (skew) symmetric
(-1) bilinear form

- For Hopf algebra $\nu_2(V) = \chi_V(\Lambda_{(1)}\Lambda_{(2)})$
- H invariant form means
$\varepsilon(h) \langle v, u \rangle = \langle h_{(1)} v, h_{(2)} u \rangle$

② $\nu_2(H) = \mathrm{Tr}(S)$

Back to Victor's question
How to differentiate $\mathrm{Rep}(Q_8)$ & $\mathrm{Rep}(D_8)$?

Here's how ---

Thm: [N-S] Let H be a s.s. Hopf algebra over $\mathbb{C}$ ① Then $\nu_n(V)$ is an invariant of the tensor category $\mathrm{Rep}(H)$ for $V \in \mathrm{Rep}(H)$, i.e. if $F: \mathrm{Rep}(H) \longrightarrow \mathrm{Rep}(K)$ is a monoidal equivalence, then
$$\nu_n(V) = \nu_n(F(V)) \text{ for all } V \in \mathrm{Rep}(H).$$

② If $F: \mathrm{Rep}(H) \longrightarrow \mathrm{Rep}(K)$, where H, K are s.s. Hopf algebras.
(Monoidal equivalence F will send regular object to regular object)
then $F(H) \cong K$, in particular,
$$\nu_n(H) = \nu_n(K)$$

③ If $\mathrm{Rep}(H)$ admits a braiding, $\nu_n(V) \in \mathbb{R}$ for all $n \in \mathbb{N}$ and $V \in \mathrm{Rep}(H)$.

Applications to the case of $\mathrm{Rep}(Q_8)$ & $\mathrm{Rep}(H_8)$

Suppose that $\mathrm{Rep}(Q_8) \overset{\text{monoidal eq.}}{\cong} \mathrm{Rep}(H_8)$ as tensor categories.
then $\nu_n(\mathbb{C}[Q_8]) \cong \nu_n(\mathbb{C}[D_8])$ $\forall\, n \geq 1$

$n=2$: $\#\{g \in Q_8 \mid x^2 = 1\} = 2$

$n=2$: $\#\{g \in D_8 \mid g^2 = 1\} = 6$

Kashima-Sommerhauser-Zhu
study $\nu_n(V)$ extensively & arrive at various generalizations

Thm: Let H be a s.s. Hopf alg and $V \in \mathrm{Rep}(H)$

$$\begin{tikzcd} \mathrm{Hom}_H(\mathbb{1}, \underbrace{H\otimes H \otimes \cdots \otimes H}_{n}) \arrow[r] \arrow[d, leftrightarrow] & \mathrm{Hom}_H(\mathbb{1}, V^{\otimes n}) \arrow[d, leftrightarrow] \\ v_1 \otimes v_2 \otimes \cdots \otimes v_n \in V^{\otimes n} \arrow[r, "\alpha"] & v_{(2)} \otimes v_{(3)} \otimes \cdots \otimes v_n \otimes v_1 \end{tikzcd}$$

$$\nu_n(V) = \mathrm{Tr}(\alpha)$$

This is a prototype to develop the Defn of Frobenius - Schur indicators categorically.

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§3 Quasitriangular Hopf algebra over any field $\mathbb{k}$

A <u>universal</u> R-matrix is an invertible element $R = \sum_i a_i \otimes b_i \in H \otimes H$ s.t.

$$\left.\begin{array}{ll} ① & \Delta^{cop}(h) = R\,\Delta(h)\,R^{-1} \\ ② & (\Delta\otimes id)R = R^{13}R^{23} \\ ③ & (id\otimes\Delta)R = R^{13}R^{12} \end{array}\right\} \Rightarrow \begin{cases} \varepsilon(a_i)b_i = 1 \\ a_i\varepsilon(b_i) = 1 \end{cases}$$

where
$$R^{12} = \textstyle\sum_i a_i \otimes b_i \otimes 1$$
$$R^{23} = \textstyle\sum_i 1 \otimes a_i \otimes b_i$$
$$R^{13} = \textstyle\sum_i a_i \otimes 1 \otimes b_i$$

A Hopf algebra H equipped with a universal R-matrix is called a Quasitriangular Hopf algebra (H,R).

If (H,R) is quasitriangular, $\mathrm{Rep}(H)$ is braided.

with braiding c given by

$$c_{U,V} : U \otimes V \to V \otimes U$$

$$u \otimes v \xrightarrow{R} \sum_i a_i u \otimes b_i v \xrightarrow{\tau} \sum_i b_i v \otimes a_i u$$

Ex: Conversely, if $\mathrm{Rep}(H)$ admits a braiding c, we get $c_{H,H} : H \otimes H \to H \otimes H$

then $R := c_{H,H}(1 \otimes 1)^{op}$

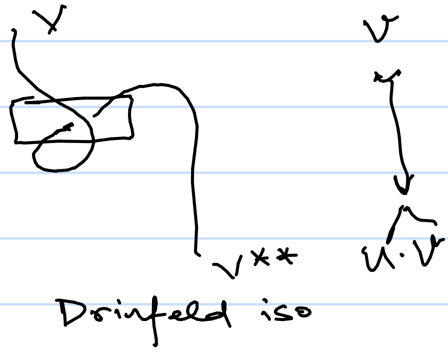
then (H, R) is Quasitriangular Hopf algebra.

Defn: Drinfeld element

Let R be a R-matrix of H.

$R = \sum a_i \otimes b_i$

Set $\underbrace{u}_{\text{Drinfeld element}} = \sum S(b_i) a_i \longleftrightarrow$



Prop:

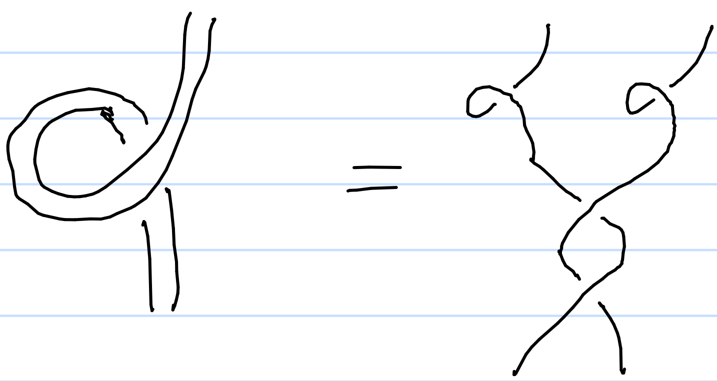
(1) $S^2(h) = u h u^{-1}$

(2) $\underbrace{u S(u)^{-1}}_{g} \in G(H)$

(3) $S^4(h) = g h g^{-1}$

(4) $\Delta u = (u \otimes u)(R^{21} R)^{-1}$

where $R^{21} = \sum b_i \otimes a_i$



Drinfeld (Quantum) double of a Hopf algebra

If H is a f.d. Hopf algebra, one can construct a Quasitriangular Hopf algebra $D(H)$, called the Drinfeld double of H, s.t.

$H \hookrightarrow D(H)$ as Hopf subalgebra

(Categorically, called as Drinfeld center construction)

Example: Let G be a finite group

$D(G) := D(\Bbbk[G])$

as a vector space, $D(G) = \Bbbk[G]^* \otimes \Bbbk[G]$ as vector space

$\{g \in G\}$ basis for $\Bbbk[G]$

$\{e(g) \in \Bbbk[G]^*\}$ its dual basis

Then • $(e(g) \otimes x) \cdot (e(h) \otimes y)$

$= \delta_{g^x, h} \; e(g) \otimes xy$

(here $g^x = x^{-1} g x$)

• $\Delta(e(g) \otimes x) = \sum_{ab=g} (e(a) \otimes x) \otimes (e(b) \otimes x)$

• $R = \sum_{g,h \in G} (e(g) \otimes h) \otimes (e(h) \otimes 1)$

• $S(e(g) \otimes x) = e((g^{-1})^x) \otimes x^{-1}$