1) Basic notions in Hopf algebra

Defn: A bialgebra A over $\mathbb{k}$ is a $\mathbb{k}$-alg with two linear maps $\Delta : A \to A \otimes A$ and $\varepsilon : A \to \mathbb{k}$

(i) (A, Δ, ε) is a coalgebra over $\mathbb{k}$

(ii) Δ, ε are algebra maps

Sweedler notation $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$

& $(\Delta \otimes I) \circ \Delta(c) = \sum c_{(1)} \otimes c_{(2)} \otimes c_{(3)} = (I \otimes \Delta) \circ \Delta(c)$

Counit property : $\sum \varepsilon(c_{(1)}) c_{(2)} = c = \sum c_{(1)} \varepsilon(c_{(2)})$

Example: G is a monoid. Then $\mathbb{k}G$ is a bialgebra over $\mathbb{k}$.

$\Delta(g) = g \otimes g \quad , \quad \varepsilon(g) = 1 \qquad \forall g \in G$

Defn: A Hopf algebra H is a bialgebra with antipode $S : H \to H$ s.t.

$S(c_{(1)}) c_{(2)} = \varepsilon(c) 1_H = c_{(1)} S(c_{(2)})$

Example: G is a group, then $\mathbb{k}G$ is a Hopf algebra with $S(g) = g^{-1} \ \forall g \in G$.

* "Antipode is unique" not additional structure

Convolution product: C is coalg, A alg $/ \mathbb{k}$ then $\mathrm{Hom}_{\mathbb{k}}(C, A)$ is an algebra under $*$

$f * g\,(c) = f(c_{(1)})\, g(c_{(2)})$

Identity is $\quad \mu(c) = \varepsilon(c) 1_A$

Ex: $\mathrm{Hom}_{\Bbbk}(H, \Bbbk) = H^*$
$\mathrm{Hom}_{\Bbbk}(H, H)$, $\mathrm{Hom}(H \otimes H, H)$
$\mathrm{Hom}(H, H \otimes H)$

By defn of S, S is the inverse of id_H in $\mathrm{Hom}(H, H)$ under $*$.
In an algebra, inverse is unique.
Thus antipode S is unique.

$m_H \in \mathrm{Hom}(H \otimes H, H)$
Then $m_H^{op} \circ S \otimes S$ and $S \circ m_H$ are right and left inverse of m_H in $\mathrm{Hom}(H \otimes H, H)$

$$\Rightarrow \quad m_H^{op} \circ (S \otimes S) = S \circ m_H$$

i.e. S is an algebra anti-homo.

Similarly, S is a coalg anti homo.
i.e. $\Delta \circ S(c) = (S \otimes S) \circ \Delta^{op}(c)$
i.e. $S(c)_{(1)} \otimes S(c)_{(2)} = S(c_{(2)}) \otimes S(c_{(1)})$

Remark: If $\varphi : H \to K$ is a bialg. homo. where H and K are Hopf algebras

(really!!)

$$S_K \circ \varphi = \varphi \circ S_H$$

(This comes automatically)

Remark: If H is a f.d. Hopf alg, H^* is a Hopf alg with
$\Delta_{H^*} : H^* \to H^* \otimes H^*$ $\qquad$ $\varepsilon_{H^*} : H^* \to \Bbbk$
$\Delta_{H^*} f(a \otimes b) = f(ab)$ $\qquad$ $f \mapsto f(1_H)$

$$S_{H^*} = S^*$$

- $H^{**} \cong H$ as Hopf algebra.

<u>Defn</u>: $\Lambda \in H \setminus \{0\}$ is called a left integral of H if $a\Lambda = \varepsilon(a)\Lambda \quad \forall\, a \in A$.

Similarly define right integral.

- A non-zero element $g \in H$ is called group like if $\Delta g = g \otimes g$

<u>Ex</u>: If g is grouplike, $S(g) = g^{-1}$ & $\varepsilon(g) = 1$
Moreover, the set of all grouplike elements of H is denoted by $G(H)$
$G(H)$ is a group under the mult. of H. $\Bbbk[G(H)]$ is a Hopf subalgebra

<u>Ex</u>: Say $\alpha \in G(H^*)$
$\Rightarrow \quad \Delta_{H^*}(\alpha) = \alpha \otimes \alpha$
but $\quad \alpha(ab) = \alpha(a)\,\alpha(b)$
$\therefore \; \alpha$ is an algebra homomorphism

In fact $G(H^*) = \{$ set of all alg homo. of $H \}$

Let H be a f.d. Hopf algebra, H^* its dual, then there are 4 actions

$H \rightharpoonup H^*$, $\quad H^* \leftharpoonup H$,
$H^* \rightharpoonup H$, $\quad H \leftharpoonup H^*$

How H acts on H^*?
$(a \rightharpoonup f^*)(b) = f(ba)$ for $a, b \in H$
$= f_{(1)}(b)\, f_{(2)}(a)$

$\therefore \quad a \rightharpoonup f = f_{(2)}(a)\, f_{(1)}$

Similarly, • $f \leftharpoonup a = f_{(1)}(a)\, f_{(2)}$

• $f \rightharpoonup a = a_{(1)}\, f(a_{(2)})$

• $a \leftharpoonup f = f(a_{(1)})\, a_{(2)}$

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2) Representation category of a Hopf algebra

Let Rep(H) be the category of f.d. representations over $\mathbb{k}$.

(i) If $V, W \in \mathrm{Rep}(H)$, then $V \otimes W \in \mathrm{Rep}(H)$ with left H-action given by

$$h \cdot (v \otimes w) = h_{(1)} \cdot v \otimes h_{(2)} \cdot w$$

(ii) If $U, V, W \in \mathrm{Rep}(H)$

$$(U \otimes V) \otimes W \xrightarrow{a} U \otimes (V \otimes W)$$

is a morphism in Rep(H)

• Pentagon, naturality follow

(iii) Let $\mathbb{1}$ be the H-module $\mathbb{k}$ with H-action given by

$$h \cdot 1_{\mathbb{k}} = \varepsilon(h)\, 1_{\mathbb{k}}$$

Now

$\mathbb{k} \otimes V \cong V \cong V \otimes \mathbb{k}$ are iso in Rep(H)

So, Rep(H) is a tensor category.
(means monoidal maybe)

(iv) Let $V \in \mathrm{Rep}(H)$

then $V^* = \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$ admits a left H-module structure given by

$$h \cdot f(v) = f(S(h) \cdot v)$$

$$\mathbb{1} \xrightarrow{db} V \otimes V^*$$

$$1_{\Bbbk} \longmapsto \sum v_i \otimes v^i$$

where $\{v_i\}, \{v^i\}$ are dual basis of V and V^*

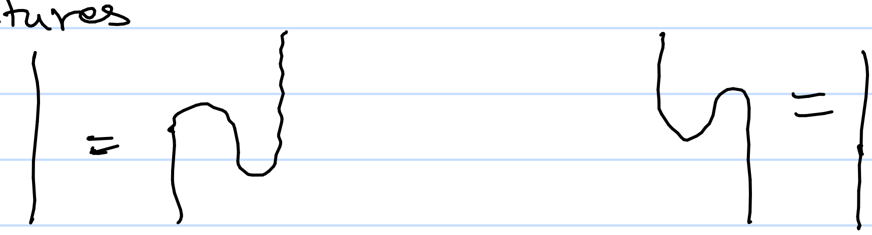
turns out $db \in \mathrm{Rep}(H)$

(So far we don't need H to be finite dimensional)

$$V^* \otimes V \xrightarrow{ev} \Bbbk \qquad \in \mathrm{Rep}(H)$$

$$\begin{tikzcd}
V \arrow[r, "\sim"] \arrow[ddrr, equal] & \Bbbk \otimes V \arrow[r, "db \otimes V"] & (V \otimes V^*) \otimes V \arrow[d, "a"] \\
& \arrow[r, phantom, "\circlearrowright"] & V \otimes (V^* \otimes V) \arrow[d, "V \otimes ev"] \\
& & V \otimes \Bbbk \arrow[l, "\sim"']
\end{tikzcd}$$

In pictures



$$\begin{tikzcd}
V^* \arrow[r, "\sim"] \arrow[d, equal] & V^* \otimes \Bbbk \arrow[r, "V^* \otimes db"] & V^* \otimes (V \otimes V^*) \arrow[d, "a"] \\
V^* & \Bbbk \otimes V^* \arrow[l, "\sim"'] \arrow[u, phantom, "\circlearrowright"] & (V^* \otimes V) \otimes V^* \arrow[l, "ev \otimes V^*"]
\end{tikzcd}$$

This also commutes

${}^{\vee}V = (V^*, db, ev)$ is a left dual of V.

- If S^{-1} exists, the right dual of V can be similarly defined where H-action is given by
$$(h\cdot f)(v) := f(S^{-1}(h)\cdot v)$$

- So, in general $\mathrm{Rep}(H)$ is a left rigid tensor category

Theorem: If H is finite dimensional, S^{-1} exists.

- So, if H is f.d., $\mathrm{Rep}(H)$ is both left & right rigid tensor category.

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3) Antipodes of Hopf algebras

§1 Hopf modules

Defn: Let $H \supseteq K$ be f.d. Hopf algebras. A right Hopf module M over (H, K) is
(i) M is a right K-module
(ii) M is a right H-comodule s.t. its coaction $\rho : M \to M \otimes_{\Bbbk} H$ is a K-module map.

Nichols-Zoeller Thm (Lagrange's Thm for Hopf algebras)

Let $H \supseteq K$ be f.d. Hopf algebras. If M is a right Hopf module over (H, K) then M is a free K-module. In particular,

H is a free K-module.
Hence, $\dim(K) \mid \dim(H)$.

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right Hopf module over (H, H) is simply called a right Hopf module over H.

Example: (Trivial Hopf module) If V is a (f-d.) vector space, then $V \otimes H$ is a Hopf module over H. The H-action and coaction are derived from the multiplication & comult. of H

Thm: (Fundamental Thm of Hopf modules)
Let M be a right Hopf module over H
then $M^{coH} \otimes H \overset{\varphi}{\cong} M$
where $M^{coH} = \{ m \in M \mid \rho(m) = m \otimes 1_H \}$

$$\varphi(m \otimes h) := m \cdot h$$

- Recall, H^* is a right Hopf module over H.
  → where the right action is denoted $\leftharpoonup$ is given by
  $f \leftharpoonup a := S(a) \rightharpoonup f$
  i.e.
  $$f \leftharpoonup a(b) = f(b S(a))$$

  → H-coaction $\rho : H^* \longrightarrow H^* \otimes H$
  $$f \longmapsto f^{(0)} \otimes f^{(1)}$$
  where $f^{(0)}, f^{(1)}$ are s.t.
  $$g * f = \sum f^{(0)} g(f^{(1)}) \text{ for all } g \in H^*.$$

Ex: H^* is a right Hopf module over H with these actions & coactions.

By the preceding thm.,
$$H^* = (H^*)^{coH} \otimes H \qquad \text{— } \circledast$$

Since H is f-d., $(H^*)^{coH}$ is 1-dim

$$(H^*)^{coH} = \{ f \in H^* \mid \rho(f) = f \otimes 1 \}$$
$$\Downarrow$$

This is just saying
$g \cdot f = f\, g(1) \;\; \forall g \in H^*$
i.e. f is a left integral of H^*

By $\circledast$ above,
the space of left integrals of H^* is 1-dimensional.

<u>Corollary</u>: $\left(\int_H^r\right) \int_H^l$ = space of (right) left integrals of H is a 1-dim space.

<u>Corollary</u>: S is bijective

<u>Proof</u>: By all the work,
$$H^* = \lambda \leftharpoonup H$$
where $\lambda \in \int_H^l$

If $S(a) = 0 \Rightarrow \lambda \leftharpoonup a = 0$

Since $(\lambda \leftharpoonup a)(b) = \lambda(b\, S(a))$
$\therefore a = 0$

hence, S is bijective.

∎

Corollary: λ is non-degenerate functional & H is a Frobenius algebra.

Proof: $H^* = \lambda \leftharpoonup H$

Then, if $\lambda(ab) = 0 \quad \forall a \in H$

$\Rightarrow \lambda \leftharpoonup S^{-1}(b) = 0$

$\Rightarrow S^{-1}(b) = 0$

$\Rightarrow b = 0$

Defn: Distinguished group like element

Let Λ be a left integral of H.

For $a \in H$, Λa is still a left integral

i.e. $\Lambda a = \Lambda \alpha(a)$ for some $\alpha \in H^*$

Exercise: $\alpha \in G(H^*)$

RADFORD'S ANTIPODE FORMULA

Let H be a f.d. Hopf algebra over $\Bbbk$ and $g \in G(H)$ and $\alpha \in G(H^*)$ be the distinguished group like element. Then

$$S^4(h) = \alpha \rightharpoonup g^{-1} h g \leftharpoonup \alpha^{-1}$$

Consequences:

By Nichols - Zoeller's Thm, $\quad (o(g) = \text{order}(g))$

$o(g), o(\alpha) \mid \dim(H)$

$\Rightarrow S^{4\dim(H)}(h) = h$

$\Rightarrow S$ has finite order

① <u>Question of Etingof</u>:

If $\Bbbk = \mathbb{C}$, then $\mathrm{Tr}(S^{2n}) = 0$ if $n \nmid \mathrm{ord}(S^2)$

<u>True</u>: when H is pointed, i.e. left H-comodules are 1-dim..

② <u>Open Question</u>: $\mathrm{ord}(S^2)$ is an invariant of $\mathrm{Rep}(H)$

i.e. If $\mathrm{Rep}(H) \cong \mathrm{Rep}(K)$, then $\mathrm{ord}(S^2_K) = \mathrm{ord}(S^2_H)$

③ In the case $\Bbbk = \mathbb{C}$

$\mathrm{tr}(S^n)$ are invariants of $\mathrm{Rep}(H)$.

True : H has Chevalley property