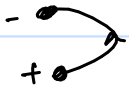
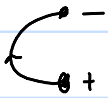
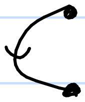
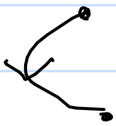
Last time we were trying to understand 2|1|0 oriented TFTs with values in Alg_2.

$$\bullet_+ \longmapsto A$$
$$\bullet_- \longmapsto A^{op}$$

 $\longmapsto {}_{A\otimes A^{op}}A_{\mathbb{R}}$

 $\longmapsto {}_{\mathbb{R}}A_{A^{op}\otimes A}$ $\Rightarrow$  $\longmapsto {}_{\mathbb{1}}A_{A\otimes A^{op}}$

Using handle cancellation

 $\longmapsto \mathrm{Hom}_{\mathbb{1}}(A,\mathbb{1})_{A\otimes A^{op}}$

 $\longmapsto \mathrm{Hom}_{A\otimes A^{op}}(A, A\otimes A^{op})_{A\otimes A^{op}}$

All three images of  should be isomorphic

$$\Rightarrow \mathrm{Hom}(A,\mathbb{1})_{A\otimes A^{op}} \simeq A_{A\otimes A^{op}}$$

(In addition ${}_{\mathbb{1}}A$ and $A_{A\otimes A^{op}}$ are f.g. projective)

this is the same as

$${}_A\mathrm{Hom}(A,\mathbb{1})_A \cong {}_AA_A \quad \text{as bimodules}$$

<u>Lemma</u>: ${}_A\mathrm{Hom}(A,\mathbb{1})_A \cong {}_AA_A$ is the same as choosing a symmetric Frobenius structure on A.

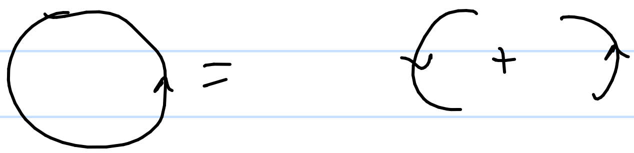
<u>Outline</u>: If $F(\bullet_+) = A$, then A is f.d separable algebra and A is given by a symm. Frobenius structure.

Theorem (Schommer-Pries)

$$\left\{\begin{array}{c}\text{Oriented 210 TFTs}\\ \text{to } \mathrm{Alg}_2\end{array}\right\} \underset{\substack{\text{as}\\ \text{2-categories}}}{\simeq} \left\{\begin{array}{c}\text{separable f.d. algebras}\\ \text{with symmetric}\\ \text{Frobenius structure}\end{array}\right\}$$

WARNING: There is no relationship between separableness and Frobeniusness (Frobenius structure is not required to be related to separable structure).

Q How does it relate to 21 TFTs?



$$= \mathrm{Hom}_{A\text{-mod-}A}(A, A) = Z(A)$$

or $\dfrac{A}{[A,A]}$

this is a commutative Frobenius algebra (as should have been the case).

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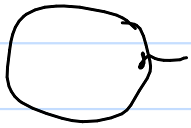
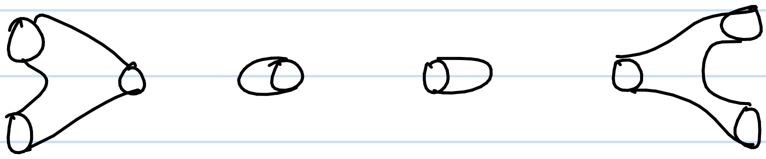
321 TFTs with values in Rex

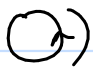
[Bartlett – Douglas – Schomer-Pries – Vicary]

We want to get generators & relations of Bord^{or}_{321} (Use Cerf Theory)

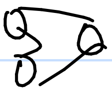
Generating object: 

Generating 1-morphisms:

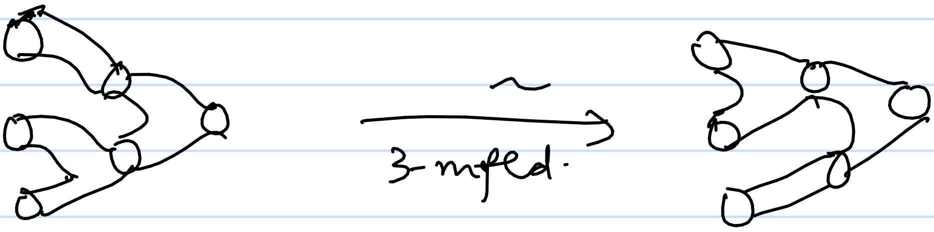


$\mathcal{F}($  $) = \mathcal{C}$

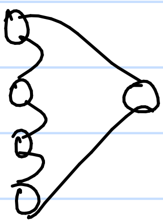
So, pair of pants give us a functor

$\mathcal{F}($  $) = \mu : \mathcal{C} \boxtimes \mathcal{C} \longrightarrow \mathcal{C}$

Generating 2-morphisms:



such 3-manifolds get sent to an associator

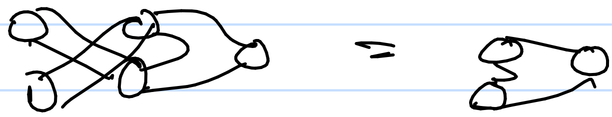
relations:  - - -

we get that the associator satisfies pentagon axiom

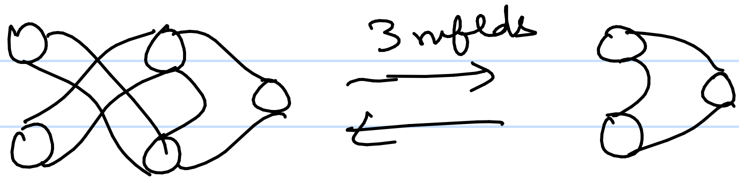
 $\longrightarrow$ gives unit

$\mathbb{1} \longrightarrow \mathcal{C}$

Before we had commutative Frob alg

because 

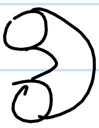
Now we have 3 mflds



this gives braiding on category $\mathcal{C}$ making it braided monoidal.

$\boxed{\text{* By dimensional reduction to a 2D TFT (by crossing with a circle) } \mathcal{C} \text{ must be finite semisimple}}$

By much harder arguments using tori, $\mathcal{C}$ is rigid.

(Existence of  only yields weak rigidity)

(Open Q: weak rigidity $\Rightarrow$ rigidity)

and $\mathcal{C}$ is modular.

- $z \mapsto \bar{z}$ its mapping cylinder gives ribbon structure.

Theorem: [B, D, S-P, V]

$$\mathrm{TFT}^{or}_{321} \simeq \text{MTC with trivial central charge}$$

Give an extended version of
$\underbrace{\text{Witten}}_{\text{gave Physics construction}}$-$\underbrace{\text{Reshetikhin-Turaev}}_{\text{gave MTC construction using surgery}}$ TFT (it is a 32-TFT)

How to go from 321 TFT to 32 TFT?

COBORDISM HYPOTHESIS

Lets think more about 210

$\bullet_+$ has a dual $= \bullet_-$ with ev/coev ⊃

But ev/coev have adjoints with unit/counit given by saddles, cups/caps + some extra stuff.

→ $F(\cdot_+)$ has to map to some object in S that has a dual

$F(ev)$ ———— some 1-mor in S that has adjoints

Defn: A symm. monoidal 2-category S is 2-dualizable (fully dualizable) if every object has a dual and every 1-mor has adjoints.

Defn: $S^{f.d}$ is the maximal fully dualizable subcategory.

(obtained by first throwing out 1-morphism which don't have adjoints iteratively, then throw out the objects without duals)

If we have
$$F: \mathrm{Bord}^{or}_{2,1,0} \longrightarrow S,$$
it must land in $S^{f.d.}$.

- $\mathrm{Alg}_2^{f.d.} = \mathrm{SepAlg}_2$

  this is fully dualizable because separable $\Rightarrow$ s.s.

  ${}_A M_B$ (A, B separable) ${}_A M$ is f.g. proj, M_B is f.g. projective

  But also needed Frobenius symmetric structure

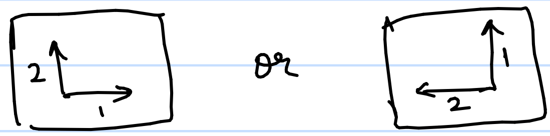
  we want to ignore this but we can't do that for oriented TFTs.

IDEA: (Baez-Dolan)
Look at framed TFTs instead

$\mathrm{Bord}_{2,1,0}^{fr}$ ~ a framing is a choice of trivialization of TM

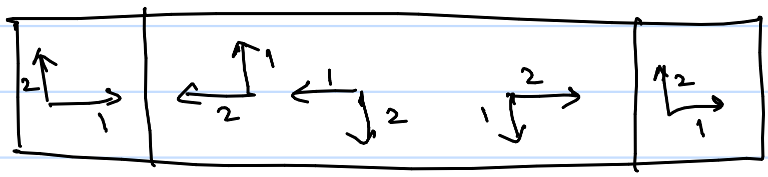
(recall: we need collars)

Framed points:



(Think of oriented TFT as framed TFT + more data)

Framed intervals:



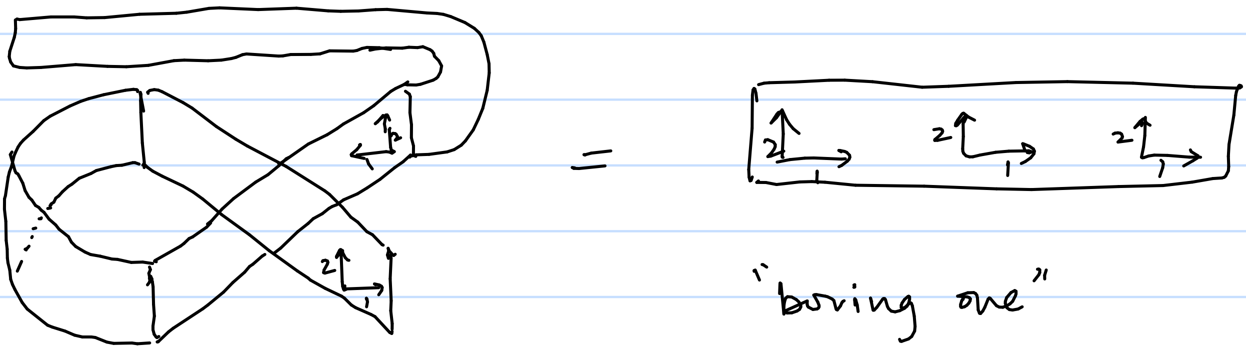
$\mathbb{Z}$ of them $\pi_1(SO(2))$

this is the evaluation map

ev

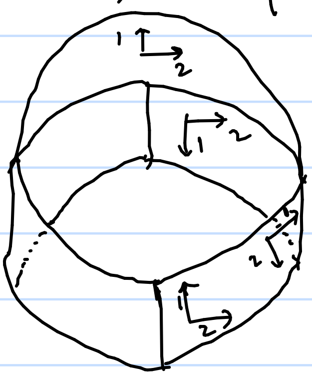
coev

lining them up gives the identity ribbon

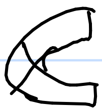
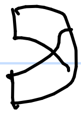


"boring one"

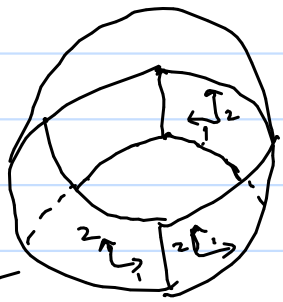
swap ∘ coev



← full clockwise rotation

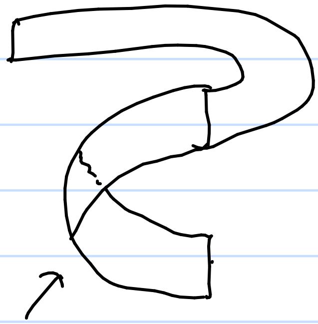
we are looking for counit of an adjunction between  and 

We can draw a version of that



rotate full turn clockwise

→ adjoint of ev is given by



turn clockwise

other adjoint goes counter clockwise once

Thus, the 3 things are different with no relation

Fill more details.

Cobordism hypothesis says:

$\mathrm{Bord}^{fr}_{2,1,0}$ is the free fully dualizable symm. mon. 2-category

Cor: $TFT^{fr}_{210}(\mathcal{S}) \simeq \mathrm{Core}(\mathcal{S}^{f.d.})$

only invertible morphisms

Cor: $TFT^{fr}_{210}(\mathrm{Alg}_2) \simeq \mathrm{Core}(\mathrm{SepAlg}_2)$

Defn: A fully extended or local TFT is a functor of symmetric mon. n-categories

$$\mathrm{Bord}_{n,n-1,\dots,0} \longrightarrow \mathcal{S}$$

Defn: Fully dualizable means objects have duals, 1-mors have adjoints, 2-mors. have adjoints - - - - (n-1)-mors. have adjoints

Thm (Lurie - Hopkins - Baez - Dolan Cobordism hypothesis)

Bord^{fr}_n is the free fully dualizable symm. mon. category.

$$TFT^{fr}_{n,\dots,0}(\mathcal{S}) \simeq \mathrm{Core}(\mathcal{S}^{f.d.})$$

FURTHER THINGS:

- There is version for oriented TFTs in terms of $SO(n)$ homotopy fixed points.

- 3210 TFTs with values in

$$TC_3 = \left\{ \begin{array}{l} \text{Tensor cats} \\ \text{bimod cats} \\ \text{bimod functors} \\ \text{bimod nat trans} \end{array} \right\}$$

Turaev-Viro framed version

(work of Douglas, S-P., Snyder)

- Relation between:

Radford's theorem $****$ $\longleftrightarrow$ $\pi_1(SO(3)) = \mathbb{Z}/2\mathbb{Z}$

Pivotal $\to$ $SO(2)$ fixed point condition
Spherical $\to$ $SO(3)$ fixed point condition