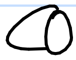
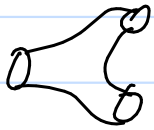
2-dim oriented TQFTs

$F(\bigcirc) = V$

 and  make V into an

$u : \mathbb{1} \to V$

associative, commutative, unital algebra

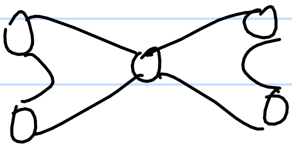
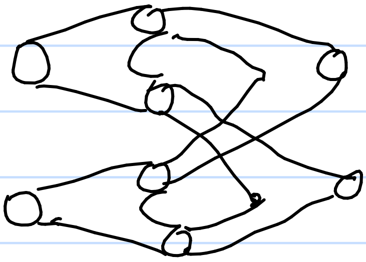
Also have  and 

$\varepsilon : V \to \mathbb{1}$ $\Delta : V \to V \otimes V$

makes V a cocomm. coassociative coalgebra

WARNING: Not a Hopf algebra. or bialgebra

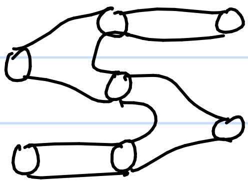
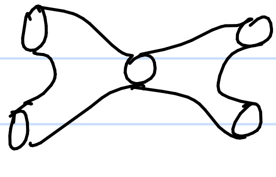
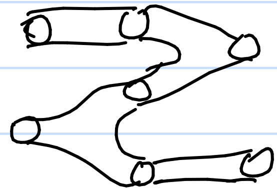
Because,

 $\neq$ 

This does not have genus

This has genus

But we have a relation

 =  = 

$(\mathrm{Id} \otimes m)(\Delta x \otimes y) = \Delta(xy) = (m \otimes \mathrm{Id})(x \otimes \Delta y)$

This is the Frobenius condition

V is a commutative Frobenius algebra.

(Does NC Frob alg correspond to some TQFT?)

Defn 1. A Frobenius algebra is an algebra and coalgebra with Frobenius condition.

Aside: Equivalent defn is
A is Frobenius if ${}_A A \cong {}_A A^*$

Thm: $\{\text{2-D TQFTs}\} \simeq \left\{\begin{matrix}\text{Commutative} \\ \text{Frobenius} \\ \text{algebras}\end{matrix}\right\}$

$\downarrow$

Morphisms are algebra + coalgebra maps

- Such maps b/w Frobenius algebras are always isomorphisms.

Defn 2: (of Frobenius algebra) It is a f.d. algebra A and a map $\tau : A \to \mathbb{1}$ s.t. the pairing $\tau(xy)$ is non-degenerate.

Example: (i) $A = \mathbb{k}$ with $\tau(1) = a \neq 0$

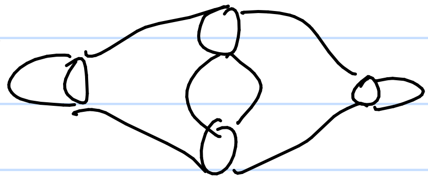
(ii) $A = \dfrac{\mathbb{k}[x]}{(x^n)}$ with $\tau(a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}) = a_{n-1}$

(iii) $A = \mathbb{k}[G]$

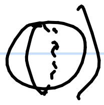
(Q If 2 TQFTs give the same invariants are they the same?)

Few examples

$\mathcal{F}\Big($  $\Big) = \tau(1)$

$\mathcal{F}\Bigg($  $\Bigg)$

$\varepsilon\big(m \circ \Delta(1)\big)$

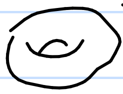
In ex (i) $\qquad \mathcal{F}\big($  $\big) = a$

basis $\{a\}$ $\qquad$ dual basis $\{\frac{1}{a}\}$

 $: \quad 1 \mapsto \frac{1}{a}(1 \otimes 1)$

$\Delta(1) = \frac{1}{a}(1 \otimes 1)$

$1 \mapsto \frac{1}{a}(1 \otimes 1 \otimes 1) \mapsto \frac{1}{a}(1 \otimes 1)$

$\therefore\ \mathcal{F}\big($  $\big) = \varepsilon\left(\mu\left(\frac{1}{a}(1 \otimes 1)\right)\right) = \varepsilon\left(\frac{1}{a}\right) = 1$

In ex (ii)

$\mathcal{F}\big($  $\big) = \tau(1) = 0$

 $: \quad 1 \mapsto \sum_{i=0}^{n-1} x^i \otimes x^{n-1-i}$

$\Delta(1) = \sum_{i=0}^{n-1} x^i \otimes x^{n-1-i}$

$1 \mapsto \sum_i 1 \otimes x^i \otimes x^{n-1-i} \mapsto \sum_{i=0}^{n-1} x^i \otimes x^{n-1-i}$

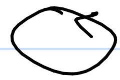
$\therefore\ \mathcal{F}\big($  $\big) = \varepsilon\left(\mu\left(\sum_{i=0}^{n-1} x^i \otimes x^{n-1-i}\right)\right) = \varepsilon(n x^{n-1})$
$= n$

$$\Delta(x^i) = (\mu \otimes id)(x^i \otimes coev)$$
$$= \sum_{j=0}^{n-1} x^{i+j} \otimes x^{n-1-j}$$

Q Think about the unoriented case.

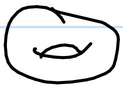
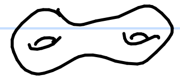
3-dim TQFTs

In 1-dim, only objects were disjoint union of $\bullet^+$ and $\bullet^-$

In 2-dim only disjoint union of 

In 3-dim disjoint union of all closed surfaces

So, we get a sequence of vector spaces

$F($  $)$, $F($  $)$, $F($  $)$ ...

Each of these has lots of structure.

- $F(\Sigma_g)$ is a representation of $MCG(\Sigma_g)$ ⇝ (not only this, the diffeomorphism group acts)

  Mapping class group

- Lots of multiplication

- Juhasz, 2017 writes down a complete description of all the above conditions.

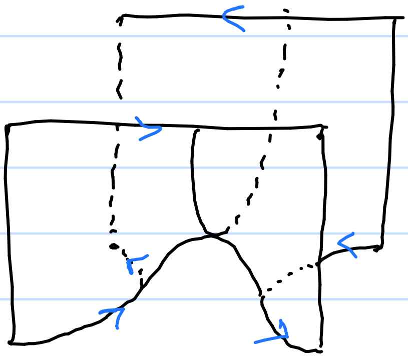
Problem: Closed surfaces are complicated.

Solution: Cut up the surfaces into

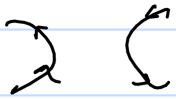
 , ..., etc.

- We now have to think about 3-dim bordisms with corners.

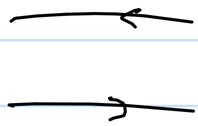
- We go back to 2-dimensions and see this idea.
- we now allow cutting up circles.

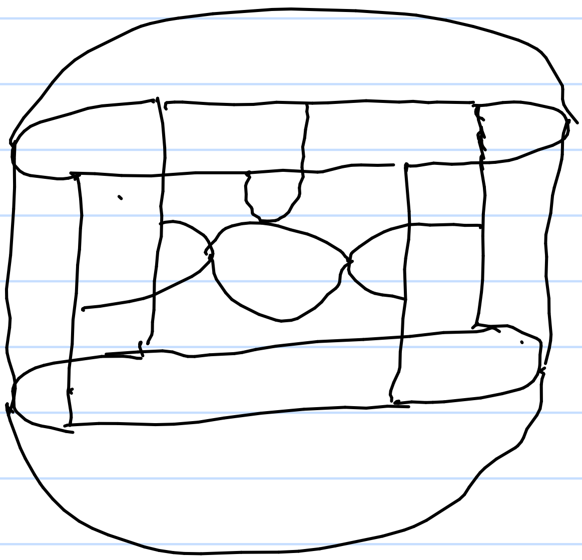


At the very bottom we have



and at top we have





← Breaking up a torus into parts using saddle, cylinder, ...

Q What is the structure that bordisms with corners have?

Ingredients:

Closed 0-manifolds

we can compose

1-dim bordisms b/w 0-manifolds

2-dim bordisms with corners b/w 1-dim bords.

can compose in 2 ways

These structures form a 2-category.

It has

Objects : $X, Y, Z, \ldots$

1-Morphisms : $F : X \to Y$

2-morphisms : Given $F : X \to Y$
$G : X \to Y$

$$\eta : F \to G$$

Can compose 1-morphisms

$$X \xrightarrow{F} Y \xrightarrow{G} Z$$

Can compose 2-morphisms in 2 ways

$$\begin{tikzcd}
X \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\eta"] \arrow[r, phantom, shift right=3ex, "\Downarrow\psi"] & Y
\end{tikzcd}$$

vertical composition

$$\begin{tikzcd}
X \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & Y \arrow[r, bend left=50, "F'"] \arrow[r, bend right=50, "G'"'] \arrow[r, phantom, "\Downarrow\eta'"] & Z
\end{tikzcd}$$

horizontal composition

WARNING: Compositions of 1-morphisms is not strictly associative.

- There is an associator and pentagon axiom is satisfied.

Examples: (1) $\mathrm{Bord}_{n,n-1,n-2}$

Objects: Closed $(n-2)$ manifolds

1-mor: $(n-1)$ bordisms

2-mor: n- bordisms with corners mod-diffeo rel. bdry

(2) Cat

Obj: Categories

1-mor: Functors

2-mor: Natural transformation

$\Bigg\}$ strict 2-category

(3)

Obj: Algebras

1-mor: bimodules

2-mor: bimodule maps

Composition is bimodules: ${}_A M_B \otimes_B {}_B M_C$

Composition of bimodule maps: (i) usual comp.

(ii) tensor of maps

(4) If $\mathcal{C}$ is a monoidal category

Objects: $*$

1-morphisms: objects of $\mathcal{C}$

compose using monoidal structure

2-morphism: morphisms of $\mathcal{C}$
(i) usual composition
(ii) tensor product

$B\mathcal{C}$ is strict $\iff$ $\mathcal{C}$ is strict

Example ③ is a generalization of of example ④.

⑤ $\Pi_2(X)$ = fundamental 2-groupoid of a space X

<u>obj</u>: pts in X
1-<u>mor</u>: paths
2-<u>mor</u>: homotopies b/w paths / homotopy