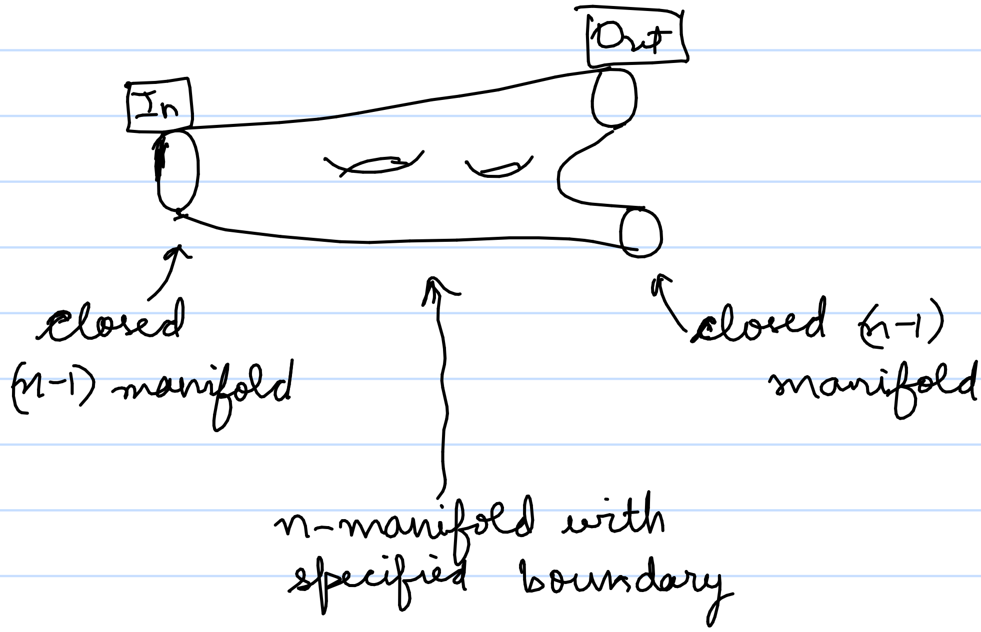
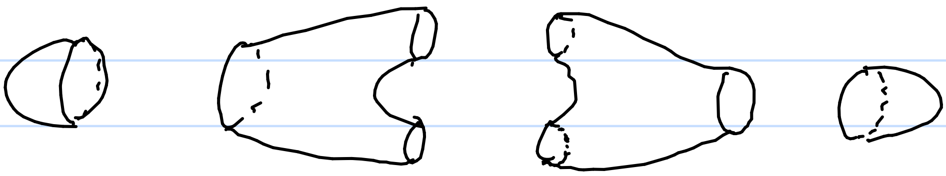
Cobordisms



Idea: Can cut a closed n-manifold up into bordisms glued together.

example



Question: What operations can you do on bordisms?

- **Gluing:** out boundary of first one matches the in boundary of second.
- **Disjoint union:** both bordisms and boundaries

Structure: $\mathrm{Bord}_n = \mathrm{Bord}_{n,n-1}$ is a category whose

Objects: closed $(n-1)$ manifolds

Morphisms: bordisms / diffeo. rel boundary

Comp: gluing

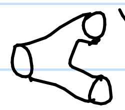
- It is a symmetric monoidal category w.r.t. to disjoint union.

- This construction has many variations
  Oriented, spin, framed
- <u>Technical problem</u>: How does gluing work?
- Solution: Use collars
  think of morphisms as having collars near boundaries

  - This fix works well for oriented TQFTs
  - Have to take care while working with framed TQFTs.

  <u>Consequence</u>: Don't frame S^1, instead frame cylinder.

TQFT: It is a symmetric monoidal functor
$$F : \mathrm{Bord}_n \longrightarrow \mathrm{Vec}$$

<u>Example</u>: $F\Big($  $\Big)$ is a linear map
$$F(0) \longrightarrow F(0) \otimes F(0)$$

F sends a closed n-manifold to a linear map from $\mathbb{R}$ to $\mathbb{R}$.
This is a number.

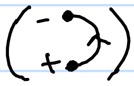
Perspectives

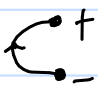
- TQFTs are numerical invariants of n-mflds which can be computed by cutting and gluing.
- TQFTs are representations of Bord_n
- TQFTs use topology to better describe algebra.

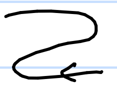
1-dimensional oriented TQFTs $\mathcal{F}: \mathrm{Bord}_1^{or} \to \mathrm{Vec}$

$$\mathcal{F}(\cdot^+) = V \quad , \quad \mathcal{F}(\cdot^-) = W$$

these two together say everything about objects.

$\mathcal{F}($  $) = \mathrm{ev} : W \otimes V \longrightarrow \mathbb{k}$

$\mathcal{F}($  $) = \mathrm{coev} : \mathbb{k} \longrightarrow V \otimes W$

$\mathcal{F}($  $) = \mathcal{F}(\leftarrow) = \mathrm{id}_W$

$\mathcal{F}($  $) = \mathcal{F}(\rightarrow) = \mathrm{id}_V$

$$\begin{tikzcd} W \arrow[r, "\mathrm{id} \otimes \mathrm{coev}"'] \arrow[rr, bend right, "\mathrm{id}_W"', "\parallel"] & W \otimes V \otimes W \arrow[r, "\mathrm{ev} \otimes \mathrm{id}"'] & W \end{tikzcd}$$

$$\begin{tikzcd} V \arrow[r, "\mathrm{coev} \otimes \mathrm{id}"] \arrow[rr, bend right, "\mathrm{id}_V"', "\parallel"] & V \otimes W \otimes V \arrow[r, "\mathrm{id} \otimes \mathrm{ev}"] & V \end{tikzcd}$$

$$\varphi : W \longrightarrow V^*$$
$$w \longmapsto ev(w, -)$$

$$\psi : V^* \longrightarrow W$$
$$v^* \longmapsto (v^* \otimes Id)(coev(1))$$

(say, $coev(1) = \sum v_i \otimes w_i$)

- $\psi \circ \varphi(w) = (ev(w,-) \otimes Id)(coev(1))$

$$= ev(w, v_i)\, w_i$$
$$= (ev \otimes id) \circ (id \otimes coev)(w)$$
$$= Id_W(w) = w$$

- $\varphi \circ \psi(v^*) = ev\big((v^* \otimes id)(coev(1)), -\big)$

$$= ev\left(\sum v^*(v_i)\, w_i, -\right)$$
$$= \sum v^*(v_i)\, ev(w_i, -)$$

$\therefore\ \varphi \circ \psi(v^*)(v) = \sum v^*(v_i)\, ev(w_i, v)$

$$= v^*\left(\sum_i v_i\, ev(w_i, v)\right)$$
$$= v^*(v)$$

$\Rightarrow \quad \varphi \circ \psi = Id$

$\Rightarrow \quad \varphi : W \longrightarrow V^*$ is an isomorphism

We get

$$\begin{tikzcd} W \otimes V \arrow[r, "ev"] \arrow[d, "\varphi \otimes I_V"'] \arrow[dr, phantom, "\circlearrowright"] & \Bbbk \\ V^* \otimes V \arrow[ur, "Ev"'] & \end{tikzcd} \qquad \begin{tikzcd} \Bbbk \arrow[r, "coev"] \arrow[dr, "Coev"'] \arrow[r, phantom, shift right=4, "\circlearrowright"] & V \otimes W \arrow[d, "I_V \otimes \varphi"] \\ & V \otimes V^* \end{tikzcd}$$

$$
\begin{tikzcd}
& V\otimes W\otimes V \arrow[dr, "I_V\otimes\varphi\otimes W"] & & V\otimes W\otimes V \arrow[dr, "I_V\otimes ev"] & \\
V \arrow[ur, "coev\otimes I_V"] \arrow[rr, "Coev\otimes I_V"'] \arrow[rrrr, bend left=60, "Id_V"] & & V\otimes V^*\otimes V \arrow[ur, "I_V\otimes\varphi^{-1}\otimes I_V"] \arrow[rr, "I_V\otimes Ev"'] & & V
\end{tikzcd}
$$

$$
\begin{tikzcd}
V \arrow[r] \arrow[rr, bend left, "Id"] \arrow[dr, dashed] & V\otimes V^*\otimes V \arrow[r] \arrow[d] & V \\
& End(V)\otimes V \arrow[ur, dashed, "T\otimes v\mapsto T(v)"'] &
\end{tikzcd}
$$

$$V\otimes V^* \longrightarrow End\, V$$

$$coev(1) \longmapsto id_V$$

$$\sum v_i\otimes v^i \longmapsto \sum v^i(-)\, v_i$$

$$v \longmapsto \underset{\sum v_i\otimes v^i}{coev(1)}\otimes v \longmapsto coev(1)(v) = \sum_i v^i(v)\, v_i = v$$

$$\Big\downarrow$$

$$\sum_i \underbrace{v^i(-)\, v_i}_{\in\, End(V)} \otimes v$$

$\therefore\ coev(1) \in End(V)$

$= Id_V$

Now $\quad coev(1) = \sum_{i=1}^{n} v_i\otimes v^i$

$\Rightarrow$ Image $(coev(1))$ is finite dimensional

Since $Id_V = coev(1)$ has finite dim image

$\Rightarrow V$ is finite dimensional

In fact, Id_V is in the image of $V\otimes V^* \longrightarrow End(V)$ iff V is finite dimensional

Thus, we see that $W =$ dual of V is actually $Hom(V, \Bbbk) = V^*$

This can be generalized to TQFTs $: \mathrm{Bord} \to \mathcal{C}$ where $\mathcal{C}$ is symmetric monoidal category with internal homs.
It turns out that for object $V \in \mathcal{C}$ with dual $V^\vee$,

$$V^\vee \cong \underline{\mathrm{Hom}}(V, \mathbb{1})$$

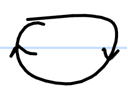
(from is similar to the one for Vect)

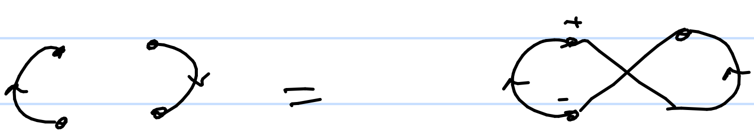
<u>Thm</u>: $\quad$ 1-dim TQFTs / nat iso $\;\cong\;$ f.d. vector spaces / iso

<u>Thm</u>: $\quad$ Category of 1-dim TQFTs $\;\underset{\text{equiv}}{\simeq}\;$ f-d- vector spaces and isomorphisms

Q: What about 1- dim unoriented TQFTs ?
(exercise)

Calculation:

$Z($  $)$



$$\mathbb{1} \xrightarrow{\text{coev}} V \otimes V^* \xrightarrow[\text{swap}]{} V^* \otimes V \xrightarrow[\text{ev}]{} \mathbb{1}$$

$$1 \longmapsto e_i \otimes e^i \longmapsto \sum e^i \otimes e_i \longmapsto \sum e^i(e_i)$$
$$= \dim(V)$$

2-dim TQFTs (oriented)

$\mathcal{F}($  $) = V$ ( is diffeo to  via mapping cylinder of $z \mapsto \bar{z}$)

Dimensional reduction:

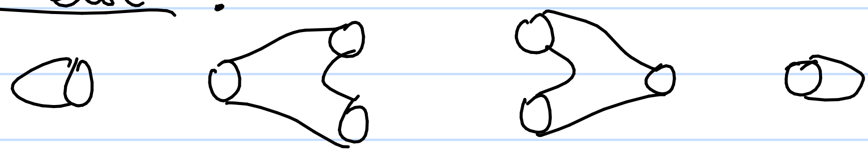
$$\mathrm{Bord}^1_{or} \xrightarrow[\times S^1]{} \mathrm{Bord}^1_{or} \xrightarrow{\mathcal{F}} \mathrm{Vec}$$

$\bullet^+ \longmapsto$  $\longmapsto V$

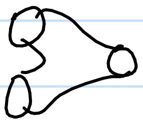
$\therefore$ V is finite dimensional

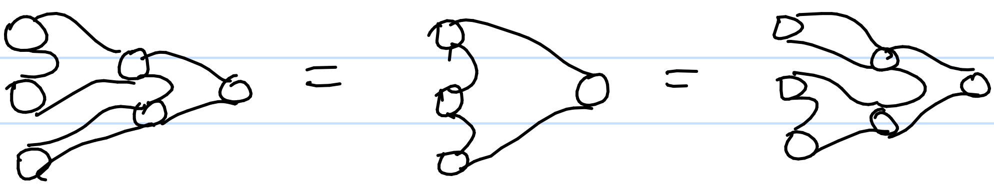
1-TQFTs obtained this way are unoriented

What else?



 $\rightsquigarrow$ $u : \mathbb{1} \to V$

 $\rightsquigarrow$ $m : V \otimes V \to V$



Associative
Unit
commutative

$\Rightarrow$ V is f.d. commutative algebra.