# HOPF ALGEBRAS AND THEIR GENERALIZATIONS FROM A CATEGORICAL POINT OF VIEW

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ABSTRACT. These lecture notes were written for a short course to be delivered in March 2017 at the *Atlantic Algebra Centre of the Memorial University of Newfoundland, Canada*.

Folklore says that (Hopf) bialgebras are distinguished algebras whose representation category admits a (closed) monoidal structure. Here we discuss generalizations of (Hopf) bialgebras based on this principle.

- The first lecture is used to present the necessary categorical background. The key notion is the lifting of functors and natural transformations to Eilenberg-Moore categories of monads.
- In the second lecture this general theory is applied to the lifting of the (closed) monoidal structure of a category to the Eilenberg-Moore category of a monad on it. This results in the notion of a *(Hopf) bimonad*.
- In the third lecture we first see how the classical structure of (Hopf) bialgebra fits this framework. The next example to be discussed is that of a *(Hopf) bialgebroid* (over an arbitrary base algebra).
- The fourth lecture is devoted to the particular (Hopf) bialgebroids whose base algebra possesses a separable Frobenius structure; known as *weak (Hopf) bialgebras*.
- The subject of the fifth lecture is *(Hopf) bimonoids* in so-called duoidal categories.

## INTRODUCTION

Since several decades, Hopf algebras have been successfully applied as symmetry objects in many different situations. For this reason they have been subject to very intensive research.

Classically, a *bialgebra* is a vector space carrying the structures both of an algebra and of a coalgebra. These are required to be compatible in the sense that the comultiplication and the counit are algebra homomorphisms; equivalently, the multiplication and the unit are coalgebra homomorphisms. A *Hopf algebra* is defined as a bialgebra with an additional property which has several equivalent formulations. The most well-known, perhaps, is the existence of a generalized inverse operation, the so-called *antipode* map.

Although Hopf algebra theory has been a highly successful and popular topic, in various applications some generalizations of Hopf algebras turned out to be needed. There are many such generalizations which apparently go in different directions.

Sometimes the underlying vector space is replaced with some more general, or simply with some different object: Hopf algebras over commutative rings, on graded vector spaces, on simplicial vector spaces, and — including all of these — even in arbitrary *braided monoidal categories* appeared in the literature.

Going even further, more general than braided monoidal categories can be taken. The categories discussed in [1] have two different, but compatible monoidal structures. In [1] they were termed 2-monoidal categories; since then (following [31]) they are more often

*Date*: March 13, 2017.

called *duoidal categories*. In this setting the algebra structure is defined in terms of one of the monoidal structures and the coalgebra structure is defined in terms of the other one.

In another direction of generalizations the axioms became weakened. In *weak Hopf algebras* [10], for example, the comultiplication is not required to preserve the unit but some weaker axioms are imposed instead.

Although the above generalizations look conceptually rather different, they share an essential feature: the structure of their category of representations. In each case it is a closed monoidal category. The aim of this course is to give a deep explanation of this fact by showing that all of the listed generalizations of Hopf algebras are instances of the unifying notion of *Hopf monad*.

Let us stress that all of the generalizations of Hopf algebras which occur in the course (and so in these notes) are strictly *associative* and strictly *coassociative*. We do not mention the generalizations known as *quasi- and coquasi Hopf algebras*. The fact is that they do not fit our framework: in all of our examples, the closed monoidal structure of the relevant representation category is *lifted* from a suitable base category. That is to say, these representation categories admit a *strictly* closed monoidal forgetful functor to this base category. This is not the case with (co)quasi Hopf algebras: although their categories of representations also admit monoidal structures, their forgetful functor to the base category is *not strictly* closed monoidal.

**Acknowledgment.** It is a pleasure to thank Yorck Sommerhäuser for organizing this mini course and for the generous invitation. The author is supported also by the Hungarian Scientific Research Fund OTKA (grant K108384).

## 1. LECTURE: LIFTING TO EILENBERG-MOORE CATEGORIES

The first lecture is used to present the necessary categorical background. The key notion is the lifting of functors and natural transformations to Eilenberg-Moore categories of monads.

In order to fix notation and terminology we recall some basic notions. For more on them we refer to [22].

**Definition 1.1.** A *category* $\mathsf{A}$ consists of

- a class of objects $X, Y, \ldots$
- for each pair of objects $X, Y$ a collection $\mathsf{A}(X,Y)$ of morphisms $X \to Y$
- for each object $X$ a map $\mathbf{1}$ from the singleton set $\mathbb{1}$ to $\mathsf{A}(X,X)$ (whose image is termed the *identity morphism* $X \to X$)
- for each triple of objects $X, Y, Z$ a map from the Cartesian product $\mathsf{A}(Y,Z) \times \mathsf{A}(X,Y)$ to $\mathsf{A}(X,Z)$ (termed the *composition*)

such that for all objects $X, Y, Z, V$ the following diagrams commute.

$$
\begin{tikzcd}
\mathsf{A}(X,Y) \arrow[r, "\mathbf{1}\times 1"] \arrow[d, "1\times\mathbf{1}"'] \arrow[dr, equal] & \mathsf{A}(Y,Y)\times\mathsf{A}(X,Y) \arrow[d, "\circ"] \\
\mathsf{A}(X,Y)\times\mathsf{A}(X,X) \arrow[r, "\circ"'] & \mathsf{A}(X,Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{A}(Z,V)\times\mathsf{A}(Y,Z)\times\mathsf{A}(X,Y) \arrow[r, "\circ\times 1"] \arrow[d, "1\times\circ"'] & \mathsf{A}(Y,V)\times\mathsf{A}(X,Y) \arrow[d, "\circ"] \\
\mathsf{A}(Z,V)\times\mathsf{A}(X,Z) \arrow[r, "\circ"'] & \mathsf{A}(X,V)
\end{tikzcd}
$$

**Examples 1.2.**

(1) The *singleton category* $\mathbb{1}$ consists of a single object and its identity morphism.

(2) In the category $\mathsf{set}$ of sets,
- the objects are sets $X, Y, \dots$
- the morphisms from $X$ to $Y$ are the maps $X \to Y$
- the identity morphism $X \to X$ is the identity map
- the composition of morphisms is the usual composition of maps.

(3) In the category $\mathsf{vec}$ of vector spaces,
- the objects are vector spaces (over a given field) $X, Y, \dots$
- the morphisms from $X$ to $Y$ are the linear maps $X \to Y$
- the identity morphism $X \to X$ is the identity map
- the composition of morphisms is the usual composition of maps.

(4) In the category $\mathsf{mod}(A)$ of modules over a given algebra $A$,
- the objects are $A$-modules $X, Y, \dots$
- the morphisms from $X$ to $Y$ are the $A$-module homomorphisms $X \to Y$
- the identity morphism $X \to X$ is the identity map
- the composition of morphisms is the usual composition of maps.

**Definition 1.3.** A *functor* $f$ from a category $\mathsf{A}$ to a category $\mathsf{B}$ consists of
- a map associating to each object $X$ of $\mathsf{A}$ an object $fX$ of $\mathsf{B}$
- for each pair of objects $X, Y$ a map $f : \mathsf{A}(X,Y) \to \mathsf{B}(fX, fY)$

such that the following diagrams commute for any objects $X, Y, Z$ of $\mathsf{A}$.

$$\begin{tikzcd} \mathbb{1} \arrow[r, "\mathbf{1}"] \arrow[d, equal] & \mathsf{A}(X,X) \arrow[d, "f"] \\ \mathbb{1} \arrow[r, "\mathbf{1}"'] & \mathsf{B}(fX,fX) \end{tikzcd} \qquad \begin{tikzcd} \mathsf{A}(Y,Z) \times \mathsf{A}(X,Y) \arrow[r, "\circ"] \arrow[d, "f\times f"'] & \mathsf{A}(X,Z) \arrow[d, "f"] \\ \mathsf{B}(fY,fZ) \times \mathsf{B}(fX,fY) \arrow[r, "\circ"'] & \mathsf{B}(fX,fZ) \end{tikzcd}$$

A functor $f : \mathsf{A} \to \mathsf{B}$ is said to be *faithful* if the induced map $\mathsf{A}(X,Y) \to \mathsf{B}(fX, fY)$ is injective for any objects $X, Y$.

The (evident) composition of functors $t : A \to B$ and $s : B \to C$ will be denoted by juxtaposition $st : A \to C$. The identity functor $A \to A$ will be denoted by 1.

**Examples 1.4.**

(1) Regarding any vector space as a plain set, and regarding linear maps as plain maps of the underlying sets we obtain a *forgetful functor* $u : \mathsf{vec} \to \mathsf{set}$.

(2) In the opposite direction, for any given field $k$ we can take the vector space $kX$ spanned by the elements of a fixed set $X$. Since any map $X \to Y$ extends to a linear map $kX \to kY$, this yields a 'linearization' functor $k : \mathsf{set} \to \mathsf{vec}$.

(3) If $A$ is an algebra over a field $k$ then every $A$-module is in particular a vector space over $k$ and $A$-module maps are in particular $k$-linear. This yields again a forgetful functor $\mathsf{mod}(A) \to \mathsf{vec}$.

(4) Let us take next algebras $A$ and $B$ over a field $k$. For a left $A$-module $V$ and a $B$-$A$ bimodule $W$, we can take the $A$-module tensor product $W \otimes_A V$, which is the quotient of the vector space $W \otimes V$ with respect to the subspace

$$\{w \cdot a \otimes v - w \otimes a \cdot v \mid a \in A,\ v \in V,\ w \in W\}.$$

Via the $B$-action on $W$, $W \otimes_A V$ is a left $B$-module, and for any left $A$-module map $h : V \to V'$ there is a left $B$-module map $1 \otimes_A h : W \otimes_A V \to W \otimes_A V'$. This defines a functor $W \otimes_A - : \mathsf{mod}(A) \to \mathsf{mod}(B)$. In particular, for any left $B$-module $W$ there is a functor $W \otimes - : \mathsf{vec} \to \mathsf{mod}(B)$ and thus for any vector space $W$ there is a functor $W \otimes - : \mathsf{vec} \to \mathsf{vec}$.

(5) In the opposite direction, for any left $B$-module $Z$ we can regard $\mathsf{mod}(B)(W,Z)$ (the set of $B$-module maps $W \to Z$) as a left $A$-module with action $(a \cdot q)(w) := q(w \cdot a)$ for a $B$-module map $q : W \to Z$, $a \in A$ and $w \in W$. Post-composition with any $B$-module map $l : Z \to Z'$ yields an $A$-module map $l \circ - : \mathsf{mod}(B)(W,Z) \to \mathsf{mod}(B)(W,Z')$ defining the functor $\mathsf{mod}(B)(W,-) : \mathsf{mod}(B) \to \mathsf{mod}(A)$. In particular, for any left $B$-module $W$ there is a functor $\mathsf{mod}(B)(W,-) : \mathsf{mod}(B) \to \mathsf{vec}$ and thus for any vector space $W$ there is a functor $\mathsf{vec}(W,-) : \mathsf{vec} \to \mathsf{vec}$.

(6) For any category $\mathsf{A}$, functors from the singleton category $\mathbb{1}$ to $\mathsf{A}$ are in a bijective correspondence with the objects of $\mathsf{A}$.

**Definition 1.5.** Consider functors $f$ and $g$ of equal source $\mathsf{A}$ and equal target $\mathsf{B}$ (we say that $f$ and $g$ are *parallel* functors). A *natural transformation* $\varphi : f \to g$ consists of

- for each object $X$ of $\mathsf{A}$ a morphism $\varphi_X : fX \to gX$ in $\mathsf{B}$

such that the following diagram commutes for any morphism $h : X \to Y$ in $\mathsf{A}$.

$$\begin{tikzcd} fX \arrow[r, "\varphi_X"] \arrow[d, "fh"'] & gX \arrow[d, "gh"] \\ fY \arrow[r, "\varphi_Y"'] & gY \end{tikzcd}$$

**Examples 1.6.**

(1) As in part (5) of Examples 1.4, there is a functor $\mathsf{mod}(B)(B,-) : \mathsf{mod}(B) \to \mathsf{mod}(B)$ for any algebra $B$. For any left $B$-module $Z$, the $B$-module map $\mathsf{mod}(B)(B,Z) \to Z$, provided by the evaluation of a $B$-module map $q : B \to Z$ on the unit element of the algebra $B$, and its inverse $Z \to \mathsf{mod}(B)(B,Z)$ sending $z$ to the map $b \mapsto b \cdot z$ define natural transformations between $\mathsf{mod}(B)(B,-) : \mathsf{mod}(B) \to \mathsf{mod}(B)$ and the identity functor.

(2) Composing the functors in parts (1) and (2) of Examples 1.4 we obtain a functor sending a set $X$ to the set of elements in the vector space $kX$. The evident inclusion maps $X \to kX$ yield the components of a natural transformation from the identity functor $\mathsf{set} \to \mathsf{set}$ to this composite functor.

(3) In the situation of part (4) of Examples 1.4, take a $B$-$A$ bimodule map $p : W \to W'$. Then for any left $A$-module $V$, the left $B$-module maps $p \otimes_A V : W \otimes_A V \to W' \otimes_A V$ yield the components of a natural transformation $W \otimes_A - \to W' \otimes_A -$.

(4) In the same setting as in the previous item (3), pre-composition with $p$ defines a left $A$-module map $- \circ p : \mathsf{mod}(B)(W',Z) \to \mathsf{mod}(B)(W,Z)$ for any left $B$-module $Z$. These are the components of a natural transformation $\mathsf{mod}(B)(W',-) \to \mathsf{mod}(B)(W,-)$.

(5) Composing of the functors in parts (4) and (5) of Examples 1.4 we obtain a functor sending a left $A$-module $V$ to the $A$-module $\mathsf{mod}(B)(W, W \otimes_A V)$. The maps $V \to \mathsf{mod}(B)(W, W \otimes_A V)$ sending $v$ to the map $w \mapsto w \otimes_A v$ yield the components of a natural transformation from the identity functor $\mathsf{mod}(A) \to \mathsf{mod}(A)$ to $\mathsf{mod}(B)(W, W \otimes_A -)$.

(6) Composing of the functors in parts (4) and (5) of Examples 1.4 in the opposite order, we obtain a functor sending a left $B$-module $Z$ to the $B$-module $W \otimes_A \mathsf{mod}(B)(W,Z)$. The evaluation maps $W \otimes_A \mathsf{mod}(B)(W,Z) \to Z$ sending $w \otimes_A q$ to $q(w)$ yield the components of a natural transformation form $W \otimes_A \mathsf{mod}(B)(W,-)$ to the identity functor $\mathsf{mod}(B) \to \mathsf{mod}(B)$.

**Exercise 1.7.** Consider functors $W \otimes_A -$ and $W' \otimes_A - : \mathsf{mod}(A) \to \mathsf{mod}(A)$ induced by $A$-bimodules $W$ and $W'$ as in part (4) of Examples 1.4. Show that any natural transformation between them is induced by an $A$-bimodule map $W \to W'$ as in part (3) of Examples 1.6.

*Hint.* To a natural transformation $\varphi : W \otimes_A - \to W' \otimes_A -$ associate the left $A$-module map

$$W \xrightarrow{\cong} W \otimes_A A \xrightarrow{\varphi_A} W' \otimes_A A \xrightarrow{\cong} W'. \tag{1.1}$$

Right multiplication by any fixed element $a$ of $A$ defines a left $A$-module map $A \to A$. The naturality of $\varphi$ with respect to it yields the commutative diagram

$$\begin{tikzcd}
W \arrow[r, "\cong"] \arrow[d, "-\cdot a"'] & W \otimes_A A \arrow[r, "\varphi_A"] \arrow[d, "1 \otimes_A - a"'] & W' \otimes A \arrow[r, "\cong"] \arrow[d, "1 \otimes_A - a"] & W' \arrow[d, "-\cdot a"] \\
W \arrow[r, "\cong"'] & W \otimes_A A \arrow[r, "\varphi_A"'] & W' \otimes A \arrow[r, "\cong"'] & W'
\end{tikzcd}$$

that is, right $A$-linearity of the map of (1.1).
Starting with an $A$-bimodule map and iterating the construction in part (3) of Examples 1.6 and the above one, we obviously re-obtain the original bimodule map. Starting with a natural transformation $\varphi : W \otimes_A - \to W' \otimes_A -$ and iterating these constructions in the opposite order, we re-obtain $\varphi$ by its naturality with respect to the left $A$-module map $A \to V$, $a \mapsto a \cdot v$, induced by an arbitrary fixed element $v$ of any left $A$-module $V$. That is, by commutativity of the following diagram.

$$\begin{tikzcd}
W \arrow[r, "\cong"] \arrow[d, "(1.1)"'] & W \otimes_A A \arrow[r, "1 \otimes_A - \cdot v"] \arrow[d, "\varphi_A"] & W \otimes_A V \arrow[d, "\varphi_V"] \\
W' \arrow[r, "\cong"'] & W' \otimes_A A \arrow[r, "1 \otimes_A - \cdot v"'] & W' \otimes_A V
\end{tikzcd}$$

**1.8. Operations with natural transformations.** For natural transformations $\varphi : f \to f'$ and $\varphi' : f' \to f''$ between parallel functors $\mathsf{A} \to \mathsf{B}$, their *composite* $\varphi' \circ \varphi : f \to f''$ has the following component at any object $X$ of $\mathsf{A}$.

$$fX \xrightarrow{\varphi_X} f'X \xrightarrow{\varphi'_X} f''X$$

The identity for this composition is the *identity natural transformation* $1 : f \to f$ with the component at the object $X$ of $\mathsf{A}$:

$$fX \xrightarrow{1} fX.$$

A natural transformation which has an inverse for this composition is termed a *natural isomorphism*. This means that each component has an inverse.

For parallel functors $f, f' : \mathsf{A} \to \mathsf{B}$ which are composable with the parallel functors $g, g' : \mathsf{B} \to \mathsf{C}$, and for natural transformations $\varphi : f \to f'$ and $\gamma : g \to g'$, there is a natural transformation $\gamma\varphi : gf \to g'f'$ — known as the *Godement product* of $\varphi$ and $\gamma$ — with component at any object $X$ of $\mathsf{A}$ occurring in the equal paths around the following diagram.

$$\begin{tikzcd} gfX \arrow[r, "\gamma_{fX}"] \arrow[d, "g\varphi_X"'] & g'fX \arrow[d, "g'\varphi_X"] \\ gf'X \arrow[r, "\gamma_{f'X}"'] & g'f'X \end{tikzcd}$$

These operations obey the following interchange law. For functors $f, f', f'' : \mathsf{A} \to \mathsf{B}$ and $g, g', g'' : \mathsf{B} \to \mathsf{C}$, and for natural transformations $\varphi : f \to f'$, $\varphi' : f' \to f''$, $\gamma : g \to g'$, $\gamma' : g' \to g''$, the equality $(\gamma' \circ \gamma)(\varphi' \circ \varphi) = (\gamma'\varphi') \circ (\gamma\varphi)$ holds.

**Example 1.9.**

(1) Part (1) of Example 1.6 describes a natural isomorphism between $\mathsf{mod}(B)(B, -)$ and the identity functor $\mathsf{mod}(B) \to \mathsf{mod}(B)$.
(2) Composing both naturally isomorphic functors in item (1) above with the forgetful functor $u : \mathsf{mod}(B) \to \mathsf{vec}$ from part (3) of Examples 1.4, we get a natural isomorphism between $\mathsf{mod}(B)(B, -) : \mathsf{mod}(B) \to \mathsf{vec}$ and $u$.

**Definition 1.10.** An *adjunction* consists of

- functors $r : \mathsf{A} \to \mathsf{B}$ and $l : \mathsf{B} \to \mathsf{A}$
- for each objects $X$ of $\mathsf{B}$ and $Y$ of $\mathsf{A}$ an isomorphism $\mathsf{A}(lX, Y) \cong \mathsf{B}(X, rY)$ which are natural in both objects $X$ and $Y$; that is, for any morphisms $p : X' \to X$ in $\mathsf{B}$ and $q : Y \to Y'$ in $\mathsf{A}$ the following diagram commutes.

$$\begin{tikzcd} \mathsf{A}(lX, Y) \arrow[r, "\cong"] \arrow[d, "q\circ - \circ lp"'] & \mathsf{B}(X, rY) \arrow[d, "rq\circ - \circ p"] \\ \mathsf{A}(lX', Y') \arrow[r, "\cong"'] & \mathsf{B}(X', rY') \end{tikzcd} \tag{1.2}$$

An adjunction is denoted by $l \dashv r : \mathsf{A} \to \mathsf{B}$ (without explicitly referring to the natural isomorphism part).

**Proposition 1.11.** *An adjunction can equivalently be described by the following data.*

- *functors* $r : \mathsf{A} \to \mathsf{B}$ *and* $l : \mathsf{B} \to \mathsf{A}$
- *natural transformations* $\eta : 1 \to rl$ *and* $\varepsilon : lr \to 1$

*such that the following diagrams commute; that is, the so-called* triangle identities *hold.*

$$\begin{tikzcd} & rlr \arrow[dr, "1\varepsilon"] & \\ r \arrow[ur, "\eta 1"] \arrow[rr, equal] & & r \end{tikzcd} \qquad \begin{tikzcd} & lrl \arrow[dr, "\varepsilon 1"] & \\ l \arrow[ur, "1\eta"] \arrow[rr, equal] & & l \end{tikzcd} \tag{1.3}$$

*The natural transformation* $\eta$ *is called the* unit *and* $\varepsilon$ *is called the* counit *of the adjunction.*

*Proof.* Suppose that isomorphisms $\xi_{X,Y} : \mathsf{A}(lX, Y) \to \mathsf{B}(X, rY)$ as in Definition 1.10 are given. The component of $\eta$ at any object $X$ of $\mathsf{B}$ is then constructed as $\xi_{X,lX}(1)$ and the component of $\varepsilon$ at any object $Y$ of $\mathsf{A}$ is constructed as $\xi^{-1}_{rY,Y}(1)$. Then the component of the upper path of the first diagram in (1.3) at any object $Y$ of $\mathsf{A}$ is equal to $r\xi^{-1}_{rY,Y}(1) \circ \xi_{rY,lrY}(1)$. By the naturality condition (1.2) this is equal to $\xi_{rY,Y}(\xi^{-1}_{rY,Y}(1))$ thus to the identity morphism

$rY \to rY$. The second triangle identity is checked symmetrically. The naturality of $\eta$ follows by applying (1.2) twice: for any morphism $t : X \to X'$ in $\mathsf{A}$,

$$rlt \circ \xi_{X,lX}(1) = \xi_{X,lX'}(lt) = \xi_{X',lX'}(1) \circ t.$$

The naturality of $\varepsilon$ is checked analogously.

Conversely, suppose that natural transformations $\eta$ and $\varepsilon$ as in the claim are given. The natural isomorphism in Definition 1.10 is constructed as

$$\mathsf{A}(lX,Y) \to \mathsf{B}(X,rY), \qquad h \mapsto rh \circ \eta_X.$$

Using the naturality of $\eta$ and $\varepsilon$ together with the triangle identities (1.3) we see that it has the inverse

$$\mathsf{B}(X,rY) \to \mathsf{A}(lX,Y), \qquad h' \mapsto \varepsilon_Y \circ lh'.$$

The commutativity of (1.2) follows by the naturality of $\eta$ and the functoriality of $r$: for any morphism $h : lX \to Y$ in $\mathsf{A}$,

$$rq \circ rh \circ \eta_X \circ p = rq \circ rh \circ rlp \circ \eta_{X'} = r(q \circ h \circ lp) \circ \eta_{X'}.$$

It is immediate to see that the above constructions are mutual inverses. □

**Example 1.12.** The natural transformations in parts (5) and (6) of Example 1.6 are the unit and the counit, respectively, of an adjunction $W \otimes_A - \dashv \mathsf{mod}(B)(W,-) : \mathsf{mod}(B) \to \mathsf{mod}(A)$.

**Exercise 1.13.** Show that whenever a functor possesses a (left or right) adjoint, it is unique up-to a natural isomorphism. In particular, the unit and the counit of an adjunction $l \dashv r$ are unique up-to a natural isomorphism.

*Hint.* Assume that both $l$ and $l'$ are left adjoints of the same functor $r$. Use the units and the counits of both adjunctions to construct mutually inverse natural isomorphisms between $l$ and $l'$. Use the forms of the triangle conditions to get inspiration.

**Definition 1.14.** A *monad* on an arbitrary category $\mathsf{A}$ consists of

- a functor $t : \mathsf{A} \to \mathsf{A}$
- a natural transformation $\eta$ from the identity functor 1 to $t$ (known as the *unit* of the monad)
- a natural transformation $\mu$ from the two fold iterate $t^2$ to $t$ (known as the multiplication of the monad)

such that the following diagrams commute.

$$\begin{tikzcd} t \arrow[r, "\eta 1"] \arrow[d, "1\eta"'] \arrow[dr, equal] & t^2 \arrow[d, "\mu"] \\ t^2 \arrow[r, "\mu"'] & t \end{tikzcd} \qquad \begin{tikzcd} t^3 \arrow[r, "\mu 1"] \arrow[d, "1\mu"'] & t^2 \arrow[d, "\mu"] \\ t^2 \arrow[r, "\mu"'] & t \end{tikzcd}$$

**Examples 1.15.**

(1) Every identity functor can be made a monad with the identity natural transformation as the multiplication and the unit.

(2) Consider an adjunction $l \dashv r : \mathsf{A} \to \mathsf{B}$. It induces a monad on $\mathsf{B}$ with functor part $rl$, unit $\eta : 1 \to rl$ of the adjunction and multiplication $1\varepsilon 1 : rlrl \to rl$ constructed from the counit $\varepsilon : lr \to 1$ of the adjunction. Indeed, associativity of the multiplication is immediate by the naturality of $\varepsilon$ and unitality follows by the triangle conditions.

(3) Consider an algebra $A$ over a field $k$ with unit map $i : k \to A$ and multiplication $m : A \otimes A \to A$. By Example 1.12 and part (2) of Examples 1.9 there is an adjunction $A \otimes - \dashv u : \mathsf{mod}(A) \to \mathsf{vec}$ whose induced monad lives on the functor $A \otimes - : \mathsf{vec} \to \mathsf{vec}$ (seen in part (4) of Examples 1.4). The unit of this monad is the natural transformation from the identity functor to $A \otimes -$ with components $i \otimes 1 : V \to A \otimes V$, for any vector space $V$. The multiplication is the natural transformation $A \otimes A \otimes - \to A \otimes -$ with components $m \otimes 1 : A \otimes A \otimes V \to A \otimes V$ for any vector space $V$.

A natural question arises at this point whether any monad is induced by an adjunction (in the way discussed in part (2) of Examples 1.15). This turns out to be the case, indeed, as we shall discuss next.

**Definition 1.16.** Consider a monad $t$ on a category $\mathsf{A}$ with unit $\eta$ and multiplication $\mu$. An *Eilenberg–Moore algebra (or module)* over this monad consists of

- an object $V$ of $\mathsf{A}$
- a morphism $v : tV \to V$ in $\mathsf{A}$ (the so-called *action*)

such that the following diagrams commute.

$$\begin{tikzcd} V \arrow[r, "\eta_V"] \arrow[dr, equal] & tV \arrow[d, "v"] \\ & V \end{tikzcd} \qquad \begin{tikzcd} t^2V \arrow[r, "\mu_V"] \arrow[d, "tv"'] & tV \arrow[d, "v"] \\ tV \arrow[r, "v"'] & V \end{tikzcd}$$

Together with the morphisms $h : V \to V'$ such that $v'.1h = h.v$, Eilenberg–Moore algebras of $t$ constitute the so-called *Eilenberg–Moore category* denoted as $\mathsf{A}^t$.

**Examples 1.17.**

(1) Regard an identity functor $1 : \mathsf{A} \to \mathsf{A}$ as a monad, in the way described in part (1) of Examples 1.15. Since its Eilenberg–Moore algebras must be unital, they must be of the form $(X, 1)$ for an any object $X$ of $\mathsf{A}$. Consequently, the Eilenberg–Moore category is $\mathsf{A}$ itself.

(2) Consider the monad in part (3) of Examples 1.15, induced by and algebra $A$. Its Eilenberg–Moore category is $\mathsf{mod}(A)$.

**1.18. The Eilenberg-Moore adjunction.** For any monad $(t, \eta, \mu)$ on $\mathsf{A}$, there is a forgetful functor $u^t : \mathsf{A}^t \to \mathsf{A}$. It sends an Eilenberg–Moore algebra $(V, v)$ to the constituent object $V$ and it acts on the morphisms as the identity map.

The forgetful functor $u^t$ has a left adjoint $f^t$ sending an object $X$ of $\mathsf{A}$ to the Eilenberg–Moore algebra $(tX, \mu_X)$ and sending a morphism $h$ to $th$.

The of unit of the adjunction should be a natural transformation $1 \to u^t f^t = t$. Its component at any object $X$ of $\mathsf{A}$ is provided by $\eta_X$ in terms of the unit $\eta$ of the monad $t$.

The counit of the adjunction should be a natural transformation $f^t u^t \to 1$. Its component at any object $(V, v)$ of $\mathsf{A}^t$ is $v$.

The monad induced by the adjunction $f^t \dashv u^t$ is equal to $(t, \eta, \mu)$.

**Theorem & Definition 1.19.** [30] *For any monads $t$ on a category $\mathsf{A}$ and $s$ on a category $\mathsf{B}$, and any functor $g : \mathsf{A} \to \mathsf{B}$, there is a bijective correspondence between the following data.*

(i) *Functors $g^\gamma : \mathsf{A}^t \to \mathsf{B}^s$ rendering commutative the following diagram.*

$$\begin{tikzcd} \mathsf{A}^t \arrow[r, "g^\gamma"] \arrow[d, "u^t"'] & \mathsf{B}^s \arrow[d, "u^s"] \\ \mathsf{A} \arrow[r, "g"'] & \mathsf{B} \end{tikzcd}$$

(ii) *Natural transformations $\gamma : sg \to gt$ such that the following diagrams commute.*

$$\begin{tikzcd} g \arrow[r, equal] \arrow[d, "\eta 1"'] & g \arrow[d, "1\eta"] \\ sg \arrow[r, "\gamma"'] & gt \end{tikzcd} \qquad \begin{tikzcd} s^2g \arrow[r, "1\gamma"] \arrow[d, "\mu 1"'] & sgt \arrow[r, "\gamma 1"] & gt^2 \arrow[d, "1\mu"] \\ sg \arrow[rr, "\gamma"'] & & gt \end{tikzcd}$$

*The functor $g^\gamma$ is termed a* lifting *of $g$ (along the* monad morphism *$\gamma$.)*

*Proof.* Assume first that the functor $g^\gamma$ in part (i) is given. It takes any Eilenberg–Moore $t$-algebra $(V,v)$ to some $s$-algebra. From the functor equality $u^s g^\gamma = gu^t$ we know that the object part of $g^\gamma(V,v)$ must be $u^s g^\gamma(V,v) = gu^t(V,v) = gV$; which carries then some $s$-action $\rho_{(V,v)} : sgV \to gV$. In particular, there is an $s$-action $\rho_{(tX,\mu_X)} : sgtX \to gtX$ for any object $X$ of $\mathsf{A}$. We construct the components of the desired natural transformation $\gamma$ as

$$\gamma_X := \ sgX \xrightarrow{sg\eta_X} sgtX \xrightarrow{\rho_{(tX,\mu_X)}} gtX$$

for any object $X$ of $\mathsf{A}$.

In order to see that it has the expected properties, note first that for any morphism $h : (V,v) \to (V',v')$ in $\mathsf{A}^t$, $u^s g^\gamma h = gu^t h = gh$ hence — since $u^s$ acts on the morphisms as the identity map — $g^\gamma h = gh$ is a morphism in $\mathsf{B}^s$ from $(gV, \rho_{(V,v)})$ to $(gV', \rho_{(V',v')})$.

For any morphism $l : X \to X'$ in $\mathsf{A}$, $f^t l = tl$ is a morphism $(tX,\mu_X) \to (tX',\mu_{X'})$ in $\mathsf{A}^t$. Hence $gtl$ is a morphism in $\mathsf{B}^s$ from $(gtX, \rho_{(tX,\mu_X)})$ to $(gtX', \rho_{(tX',\mu_{X'})})$. Using this, and the naturality of the unit $\eta$ of the monad $t$, we see that $\gamma$ is natural; that is, the following diagram commutes for any morphism $l : X \to X'$ in $\mathsf{A}$.

$$\begin{tikzcd} sgX \arrow[r, "sg\eta_X"] \arrow[d, "sgl"'] & sgtX \arrow[r, "\rho_{(tX,\mu_X)}"] \arrow[d, "sgtl"] & gtX \arrow[d, "gtl"] \\ sgX' \arrow[r, "sg\eta_{X'}"'] & sgtX' \arrow[r, "\rho_{(tX',\mu_{X'})}"'] & gtX' \end{tikzcd}$$

The compatibility of $\gamma$ with the units of both monads $t$ and $s$; that is, commutativity of the following diagram follows by the unitality of the $s$-action $\rho_{(tX,\mu_X)}$ and the naturality of the unit $\eta$ of $s$.

$$\begin{tikzcd} gX \arrow[r, "g\eta_X"] \arrow[d, "\eta_{gX}"'] & gtX \arrow[d, "\eta_{gtX}"] \arrow[dr, equal, bend left] & \\ sgX \arrow[r, "sg\eta_X"'] & sgtX \arrow[r, "\rho_{(tX,\mu_X)}"'] & gtX \end{tikzcd}$$

Finally we should see the compatibility of $\gamma$ with the multiplications of both monad $t$ and $s$. For this purpose note that the associativity of the multiplication $\mu$ of the monad $t$ makes

$\mu_X : t^2X \to tX$ a morphism in $\mathsf{A}^t$, from $(t^2X, \mu_{tX})$ to $(tX, \mu_X)$, for any object $X$ of $\mathsf{A}$. Hence $g\mu_X$ is a morphism in $\mathsf{B}^s$ from $(gt^2X, \rho_{(t^2X,\mu_{tX})})$ to $(gtX, \rho_{(tX,\mu_X)})$. Using this, the unitality of $\mu$, the associativity of the action $\rho_{(tX,\mu_X)}$ and the naturality of $\mu$, we see that the following diagram — expressing the compatibility of $\gamma$ with the multiplications of $t$ and $s$ — commutes.

$$\begin{tikzcd}
s^2gX \arrow[r, "s^2g\eta_X"] \arrow[dd, "\mu_{gX}"'] & s^2gtX \arrow[r, "s\rho_{(tX,\mu_X)}"] \arrow[dd, "\mu_{gtX}"] & sgtX \arrow[r, "sg\eta_{tX}"] \arrow[dr, equal] & sgt^2X \arrow[r, "\rho_{(t^2X,\mu_{tX})}"] \arrow[d, "sg\mu_X"] & gt^2X \arrow[dd, "g\mu_X"] \\
 & & & sgtX \arrow[dr, "\rho_{(tX,\mu_X)}"'] & \\
sgX \arrow[r, "sg\eta_X"'] & sgtX \arrow[rrr, "\rho_{(tX,\mu_X)}"'] & & & gtX
\end{tikzcd}$$

In the opposite direction, assume that the monad morphism $\gamma$ in part (ii) is given. We construct the desired functor $g^\gamma$ sending an object $(V, v)$ of $\mathsf{A}^t$ to

$$\Big(gV,\ sgV \xrightarrow{\gamma_V} gtV \xrightarrow{gv} gV\ \Big)$$

and sending a morphism $h : (V, v) \to (V', v')$ to $gh$.

Unitality of the action $gv \circ \gamma_V : sgV \to gV$ follows by the unitality of $v$ and the compatibility of $\gamma$ with the units of $t$ and $s$:

$$\begin{tikzcd}
gV \arrow[r, equal] \arrow[d, "\eta_{gV}"'] & gV \arrow[d, "g\eta_V"] \arrow[dr, equal, bend left] & \\
sgV \arrow[r, "\gamma_V"'] & gtV \arrow[r, "gv"'] & gV
\end{tikzcd}$$

and its associativity follows by the naturality of $\gamma$, the associativity of $v$ and the compatibility of $\gamma$ with the multiplications of $t$ and $s$:

$$\begin{tikzcd}
s^2gV \arrow[r, "s\gamma_V"] \arrow[dd, "\mu_{gV}"'] & sgtV \arrow[r, "sgv"] \arrow[d, "\gamma_{tV}"] & sgV \arrow[d, "\gamma_V"] \\
 & gt^2V \arrow[r, "gtv"] \arrow[d, "g\mu_V"] & gtV \arrow[d, "gv"] \\
sgV \arrow[r, "\gamma_V"'] & gtV \arrow[r, "gv"'] & gV.
\end{tikzcd}$$

This proves that $(gV, gv \circ \gamma_V)$ is an object of $\mathsf{B}^s$. Also $gh$ is a morphism in $\mathsf{B}^s$; that is, the following diagram commutes, by the naturality of $\gamma$ and since $h$ is a morphism in $\mathsf{A}^t$:

$$\begin{tikzcd}
sgV \arrow[r, "\gamma_V"] \arrow[d, "sgh"'] & gtV \arrow[r, "gv"] \arrow[d, "gth"] & gV \arrow[d, "gh"] \\
sgV' \arrow[r, "\gamma_{V'}"'] & gtV' \arrow[r, "gv'"'] & gV'
\end{tikzcd}$$

The above constructions are mutual inverses. Indeed, staring with a functor $g^\gamma$ as in part (i), and iterating both constructions, we arrive at the functor sending and Eilenberg–Moore

$t$-algebra $(V,v)$ to the pair consisting of the object $gV$ and the $s$-action occurring in the equal paths around the commutative diagram

$$\begin{tikzcd} sgV \arrow[r, "sg\eta_V"] \arrow[dr, equal, bend right] & sgtV \arrow[r, "\rho_{(tV,\mu_V)}"] \arrow[d, "sgv"] & gtV \arrow[d, "gv"] \\ & sgV \arrow[r, "\rho_{(V,v)}"'] & gV \end{tikzcd}$$

That is, we re-obtain the functor $g^\gamma$. Here we used that — by its associativity — $v$ is a morphism in $\mathsf{A}^t$ from $(tV,\mu_V)$ to $(V,v)$ and hence $gv$ is a morphism in $\mathsf{B}^s$ from $(gtV,\rho_{(tV,\mu_V)})$ to $(gV,\rho_{(V,v)})$; as well as the unitality of $v$.

Starting with a monad morphism $\gamma$ as in part (ii) and iterating both constructions in the opposite order, we arrive at the natural transformation with component at an object $X$ of $\mathsf{A}$ occurring in in the equal paths around the commutative diagram

$$\begin{tikzcd} sgX \arrow[r, "\gamma_X"] \arrow[d, "sg\eta_X"'] & gtX \arrow[d, "gt\eta_X"] \arrow[dr, equal, bend left] & \\ sgtX \arrow[r, "\gamma_{tX}"'] & gt^2X \arrow[r, "g\mu_X"'] & gtX \end{tikzcd}$$

That is, we re-obtain $\gamma$. Here we used the naturality of $\gamma$ and the unitality of $t$. ☐

**Example 1.20.** As in part (4) of Examples 1.4, any vector space $W$ over a field $k$ induces a functor $W\otimes - : \mathsf{vec} \to \mathsf{vec}$. We want to lift this functor. To this end, consider the following two monads on $\mathsf{vec}$. The first one be the identity functor as in part (1) of Examples 1.15, with Eilenberg–Moore category $\mathsf{vec}$, see part (1) of Examples 1.17. The second one be the monad in part (3) of Examples 1.15, induced on $\mathsf{vec}$ by a $k$-algebra $A$, with Eilenberg–Moore category $\mathsf{mod}(A)$ in part (2) of Examples 1.17.

We are interested in lifted functors

$$\begin{tikzcd} \mathsf{vec} \arrow[r, dashed] \arrow[d, equal] & \mathsf{mod}(A) \arrow[d, "u"] \\ \mathsf{vec} \arrow[r, "W\otimes -"'] & \mathsf{vec} \end{tikzcd}$$

By Theorem & Definition 1.19, they correspond bijectively to monad morphisms with components $\gamma_X : A\otimes W\otimes X \to W\otimes X$, for any vector space $X$.

Any element $x$ of a vector space $X$ induces a linear map $k\to X$, sending the multiplicative unit 1 of $k$ to the chosen element $x$. Hence naturality of $\gamma$ implies that

$$\gamma_X(a\otimes w\otimes x) = \gamma_k(a\otimes w)\otimes x, \qquad \forall x\in X,\ w\in W,\ a\in A.$$

Moreover, $\gamma$ obeys the compatibility conditions of a monad morphism if and only if $\gamma_k : A\otimes W \to W$ is a unital and associative $A$-action on $W$.

In a word, liftings of the functor $W\otimes - : \mathsf{vec}\to\mathsf{vec}$ to $\mathsf{vec}\to\mathsf{mod}(A)$ are in a bijective correspondence with the $A$-actions on $W$.

**Exercise 1.21.** Consider monads $t$ on a category $\mathsf{A}$, $s$ on $\mathsf{B}$ and $z$ on $\mathsf{C}$; together with functors $f : \mathsf{A}\to\mathsf{B}$ and $g : \mathsf{B}\to\mathsf{C}$. Assume that they admit liftings $f^\varphi : \mathsf{A}^t \to \mathsf{B}^s$ and $g^\gamma : \mathsf{B}^s \to \mathsf{C}^z$

along some monad morphisms $\varphi$ and $\gamma$. By the commutativity of the diagram

$$\begin{tikzcd}
\mathsf{A}^t \arrow[r, "f^\varphi"] \arrow[d, "u^t"'] & \mathsf{B}^s \arrow[r, "g^\gamma"] \arrow[d, "u^s"] & \mathsf{C}^z \arrow[d, "u^z"] \\
\mathsf{A} \arrow[r, "f"'] & \mathsf{B} \arrow[r, "g"'] & \mathsf{C}
\end{tikzcd}$$

we know that $g^\gamma f^\varphi$ is a lifting of $gf$. Compute the corresponding monad morphism.

*Hint.*

$$zgf \xrightarrow{\gamma 1} gsf \xrightarrow{1\varphi} gft$$

**Theorem & Definition 1.22.** [30] *For any monads $t$ on a category $\mathsf{A}$ and $s$ on a category $\mathsf{B}$, let $h,g:\mathsf{A}\to\mathsf{B}$ be functors admitting liftings $h^\chi, g^\gamma:\mathsf{A}^t\to\mathsf{B}^s$ (along respective monad morphisms $\chi$ and $\gamma$). Then for any natural transformation $\omega: h\to g$ the following assertions hold.*

(1) *There exists at most one natural transformation $\overline{\omega}: h^\chi \to g^\gamma$ such that $u^s\overline{\omega}_{(V,v)} = \omega_V$ for all Eilenberg–Moore $t$-algebras $(V,v)$.*
(2) *The natural transformation $\overline{\omega}$ in part (1) exists if and only if the following diagram commutes.*

$$\begin{tikzcd}
sh \arrow[r, "\chi"] \arrow[d, "1\omega"'] & ht \arrow[d, "\omega 1"] \\
sg \arrow[r, "\gamma"'] & gt
\end{tikzcd}$$

*The natural transformation $\overline{\omega}$ — provided that it exists — is called the* lifting *of $\omega$.*

*Proof.* Part (1) is immediate from the faithfulness of $u^s$.

(2) The lifting $\overline{\omega}$ exists if and only if $\omega_V: hV\to gV$ is a morphism in $\mathsf{B}^s$; that is, the following diagram commutes.

$$\begin{tikzcd}
shV \arrow[r, "\chi_V"] \arrow[d, "s\omega_V"'] & htV \arrow[r, "hv"] & hV \arrow[d, "\omega_V"] \\
sgV \arrow[r, "\gamma_V"'] & gtV \arrow[r, "gv"'] & gV
\end{tikzcd} \tag{1.4}$$

If the diagram of part (1) commutes then so does (1.4) by the naturality of $\omega$. Conversely, if (1.4) commutes then so does

$$\begin{tikzcd}
shX \arrow[rr, "\chi_X"] \arrow[dr, "sh\eta_X"] \arrow[ddd, "s\omega_X"'] & & htX \arrow[d, "ht\eta_X"] \arrow[dr, equal, bend left] & \\
& shtX \arrow[r, "\chi_{tX}"] \arrow[d, "s\omega_{tX}"'] & ht^2X \arrow[r, "h\mu_X"] \arrow[d, phantom, "(1.4)"] & htX \arrow[d, "\omega_{tX}"] \\
& sgtX \arrow[r, "\gamma_{tX}"] & gt^2X \arrow[r, "g\mu_X"] & gtX \arrow[dl, equal, bend left] \\
sgX \arrow[rr, "\gamma_X"'] \arrow[ur, "sg\eta_X"'] & & gtX \arrow[u, "gt\eta_X"'] &
\end{tikzcd}$$

for any object $X$ of $\mathsf{A}$, by the naturality of $\chi$, $\gamma$ and $\omega$ and by the unitality of $t$. □

**Example 1.23.** By part (3) of Examples 1.6, any linear map $p : W \to W'$ induces a natural transformation $p \otimes - : W \otimes - \to W' \otimes -$ between functors $\mathsf{vec} \to \mathsf{vec}$.

If $W$ and $W'$ are modules over an algebra $A$, then the induced functors $W \otimes -$ and $W' \otimes -$ lift to $\mathsf{vec} \to \mathsf{mod}(A)$.

Theorem & Definition 1.22 says that the natural transformation $p \otimes - : W \otimes - \to W' \otimes -$ between functors $\mathsf{vec} \to \mathsf{vec}$ lifts to a natural transformation between the lifted functors $\mathsf{vec} \to \mathsf{mod}(A)$ if and only if $p$ is an $A$-module map.

**Theorem & Definition 1.24.** [15, Theorem 3.13] *Consider a monad $t$ on a category $\mathsf{B}$ (with unit $\eta^t$ and multiplication $\mu^t$) and a monad $s$ on $\mathsf{A}$ (with unit $\eta^s$ and multiplication $\mu^s$). Take an adjunction $l \dashv r : \mathsf{A} \to \mathsf{B}$ (with unit $\eta$ and counit $\varepsilon$) such that $l$ admits a lifting $l^\lambda : \mathsf{B}^t \to \mathsf{A}^s$ along some monad morphism $\lambda : sl \to lt$. The following assertions are equivalent.*

(i) *Also $r$ admits a lifting $r^\rho : \mathsf{A}^s \to \mathsf{B}^t$ such that $l^\lambda \dashv r^\rho$ is an adjunction whose unit is the lifting of $\eta$ and the counit is the lifting of $\varepsilon$.*
(ii) *$\lambda$ is invertible.*

*In this situation the adjunction $l^\lambda \dashv r^\rho$ is said to be the* lifting *of $l \dashv r$.*

*Proof.* By Theorem & Definition 1.19 and Theorem & Definition 1.22 assertion (i) is equivalent to the existence of a monad morphism $\rho : tr \to rs$ rendering commutative the following diagrams.

$$\begin{tikzcd} t \arrow[rr, equal] \arrow[d, "1\eta"'] & & t \arrow[d, "\eta 1"] \\ trl \arrow[r, "\rho 1"'] & rsl \arrow[r, "1\lambda"'] & rlt \end{tikzcd} \qquad \begin{tikzcd} slr \arrow[r, "\lambda 1"] \arrow[d, "1\varepsilon"'] & ltr \arrow[r, "1\rho"] & lrs \arrow[d, "\varepsilon 1"] \\ s \arrow[rr, equal] & & s \end{tikzcd} \tag{1.5}$$

(The bottom row of the first diagram and the top row of the second diagram are computed as in Exercise 1.21.) Use the naturality of $\eta$ and $\varepsilon$ together with the triangle conditions to see that the adjunction $l \dashv r$ gives rise to mutually inverse bijections between the following families of natural transformations.

$$\begin{aligned} \Phi \quad &: \mathsf{nat}(tr, rs) \to \mathsf{nat}(lt, sl), \qquad \xi \mapsto lt \xrightarrow{11\eta} ltrl \xrightarrow{1\xi 1} lrsl \xrightarrow{\varepsilon 11} sl \\ \Phi^{-1} \quad &: \mathsf{nat}(lt, sl) \to \mathsf{nat}(tr, rs), \qquad \zeta \mapsto tr \xrightarrow{\eta 11} rltr \xrightarrow{1\zeta 1} rslr \xrightarrow{11\varepsilon} rs \end{aligned}$$

($\xi$ and $\Phi(\xi)$ are called *mates* under the adjunction). The conditions of (1.5) are equivalent to the statement that $\lambda$ and $\Phi(\rho)$ are mutual inverses. Indeed, apply the functor $l$ to the equal paths of the first diagram of (1.5); and post-compose the resulting equal expressions with $\varepsilon 11$. Use a triangle condition to deduce $\lambda\Phi(\rho) = 1$. Conversely, apply $r$ to both sides of the equality $\lambda\Phi(\rho) = 1$ and pre-compose the resulting equal expressions with $\eta 1$. Use a triangle condition to deduce commutativity of the first diagram of (1.5). By symmetric steps commutativity of the second diagram of (1.5) is shown to be equivalent to $\Phi(\rho)\lambda = 1$.

This shows that whenever (i) holds, $\lambda$ is invertible as stated in (ii).

Conversely, if $\lambda$ is invertible as in (ii), then we can construct $\rho := \Phi^{-1}(\lambda^{-1})$. It will render commutative the diagrams of (1.5) by the above considerations and it will be a monad morphism by commutativity of the following diagrams (what follows using naturality of the occurring morphisms, the fact that $\lambda$ is a monad morphism and the triangle conditions on $\eta$ and $\varepsilon$).

$$
\begin{tikzcd}
r \arrow[rrr, equal] \arrow[r, "\eta 1"] \arrow[d, "\eta^t 1"'] & rlr \arrow[r, equal] \arrow[d, "11\eta^t 1"'] & rlr \arrow[r, "1\varepsilon"] \arrow[d, "1\eta^s 11"] & r \arrow[d, "1\eta^s"] \\
tr \arrow[r, "\eta 11"'] & rltr \arrow[r, "1\lambda^{-1}1"'] & rslr \arrow[r, "11\varepsilon"'] & rs
\end{tikzcd}
$$

$$
\begin{tikzcd}
t^2r \arrow[rrr, "1\eta 11"] \arrow[dr, "\eta 111"] \arrow[dddd, "\mu^t 1"'] & & & trltr \arrow[r, "11\lambda^{-1}1"] & trslr \arrow[r, "111\varepsilon"] & trs \arrow[d, "\eta 111"] \\
 & rlt^2r \arrow[rr, "111\eta 11"] \arrow[dr, "1\lambda^{-1}11"] \arrow[ddd, "11\mu^t 1"'] & & rltrltr \arrow[r, "1111\lambda^{-1}1"] & rltrslr \arrow[r, "11111\varepsilon"] & rltrs \arrow[d, "1\lambda^{-1}11"] \\
 & & rsltr \arrow[r, "111\eta 11"] \arrow[dr, equal] & rslrltr \arrow[r, "1111\lambda^{-1}1"] \arrow[d, "11\varepsilon 111"] & rslrslr \arrow[r, "11111\varepsilon"] & rslrs \arrow[d, "11\varepsilon 1"] \\
 & & & rsltr \arrow[r, "11\lambda^{-1}1"] & rs^2lr \arrow[r, "111\varepsilon"] \arrow[d, "1\mu^s 11"] & rs^2 \arrow[d, "1\mu^s"] \\
tr \arrow[r, "\eta 11"'] & rltr \arrow[rrr, "1\lambda^{-1}1"'] & & & rslr \arrow[r, "11\varepsilon"'] & rs
\end{tikzcd}
$$

Thus the assertions of (i) hold. □

## 2. Lecture: (Hopf) bimonads

In the second lecture the general theory introduced in the first lecture is applied to the lifting of the (closed) monoidal structure of a category to the Eilenberg-Moore category of a monad on it. This results in the notion of a *(Hopf) bimonad*. Our key references here are [23], [21] and [15].

**Definition 2.1.** A *monoidal structure* on a category $\mathsf{A}$ consists of

- a distinguished object $I$, called the *monoidal unit* (regarded as a functor from the singleton category to $\mathsf{A}$)
- a functor $\otimes$ from the Cartesian product category $\mathsf{A} \times \mathsf{A}$ to $\mathsf{A}$ (called the *monoidal product*)
- natural isomorphisms $\alpha : (-\otimes -)\otimes - \to -\otimes(-\otimes -)$ (called the *associativity constraint*), $\lambda : I \otimes - \to 1$ and $\rho : -\otimes I \to 1$ (called the *unit constraints*)

such that for all objects $X, Y, Z, V$, the following diagrams commute.

$$
\begin{tikzcd}
((X\otimes Y)\otimes Z)\otimes V \arrow[r, "\alpha_{X\otimes Y,Z,V}"] \arrow[d, "\alpha_{X,Y,Z}\otimes 1"'] & (X\otimes Y)\otimes(Z\otimes V) \arrow[r, "\alpha_{X,Y,Z\otimes V}"] & X\otimes(Y\otimes(Z\otimes V)) \\
(X\otimes(Y\otimes Z))\otimes V \arrow[rr, "\alpha_{X,Y\otimes Z,V}"'] & & X\otimes((Y\otimes Z)\otimes V) \arrow[u, "1\otimes\alpha_{Y,Z,V}"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
(X\otimes I)\otimes Y \arrow[rr, "\alpha_{X,I,Y}"] \arrow[dr, "\rho_X\otimes 1"'] & & X\otimes(I\otimes Y) \arrow[dl, "1\otimes\lambda_Y"] \\
 & X\otimes Y &
\end{tikzcd}
$$

They are known as the *pentagon* and *triangle conditions*, respectively.

**Examples 2.2.**

(1) The category of sets is monoidal via the monoidal product provided by the Cartesian product and monoidal unit the singleton set.
(2) The category of vector spaces is monoidal via the monoidal product provided by the tensor product of vector spaces and monoidal unit the base field.
(3) The category of bimodules over any algebra $A$ is monoidal via the monoidal product provided by the $A$-module tensor product and monoidal unit the regular $A$-bimodule $A$ (with actions given by the multiplication).
(4) For any category $\mathsf{A}$, the category whose objects are the endofunctors $\mathsf{A} \to \mathsf{A}$, and whose morphisms are the natural transformations, admits the following monoidal structure. The monoidal product is the composition of functors (and the induced Godement product on the natural transformations). The monoidal unit is the identity functor. The associativity and unit constraints are identity natural transformations.

**Exercise 2.3.** Verify the commutativity of the following diagrams in an arbitrary monoidal category, for arbitrary objects $X, Y$.

$$\begin{tikzcd}
(X\otimes Y)\otimes I \arrow[rr, "\alpha_{X,Y,I}"] \arrow[dr, "\rho_{X\otimes Y}"'] & & X\otimes(Y\otimes I) \arrow[dl, "1\otimes\rho_Y"] \\
& X\otimes Y &
\end{tikzcd}
\qquad
\begin{tikzcd}
(I\otimes X)\otimes Y \arrow[rr, "\alpha_{I,X,Y}"] \arrow[dr, "\lambda_X\otimes 1"'] & & I\otimes(X\otimes Y) \arrow[dl, "\lambda_{X\otimes Y}"] \\
& X\otimes Y &
\end{tikzcd}$$

*Hint.* For the first diagram, take the pentagon condition for $Z = V = I$. Post-compose the top path with

$$X\otimes(Y\otimes(I\otimes I)) \xrightarrow{1\otimes(1\otimes\lambda_I)} X\otimes(Y\otimes I) \xrightarrow{\alpha^{-1}_{X,Y,I}} (X\otimes Y)\otimes I \xrightarrow{\rho_{X\otimes Y}} X\otimes Y. \tag{2.1}$$

Use the naturality of $\alpha$, the triangle condition and the naturality of $\rho$ to see that the resulting morphism is equal to

$$((X\otimes Y)\otimes I)\otimes I \xrightarrow{\rho_{(X\otimes Y)\otimes I}} (X\otimes Y)\otimes I \xrightarrow{\rho_{X\otimes Y}} X\otimes Y.$$

Post-compose also the other path of the pentagon condition for $Z = V = I$ with (2.1). Use now first the triangle condition, and then the naturality of $\alpha$ and $\rho$ to see that the resulting morphism is equal to

$$((X\otimes Y)\otimes I)\otimes I \xrightarrow{\rho_{(X\otimes Y)\otimes I}} (X\otimes Y)\otimes I \xrightarrow{\alpha_{X,Y,I}} X\otimes(Y\otimes I) \xrightarrow{1\otimes\rho_Y} X\otimes Y.$$

Since $\rho : ((X\otimes Y)\otimes I)\otimes I \to (X\otimes Y)\otimes I$ is invertible, this proves the commutativity of the first diagram. The second diagram is handled symmetrically.

**2.4. Coherence.** Mac Lane's Coherence Theorem [22, Section VII.2] asserts the following. In any monoidal category, taking composites of monoidal products of the associativity and the unit constraints and of identity morphisms, one can construct at most one morphism with a given domain and codomain. That is to say, such morphisms are uniquely characterized by their domain and codomain.

By this reason, for brevity, we shall not explicitly denote the associativity and the unit constraints. That is, we write $X$ both instead of $I\otimes X$ and $X\otimes I$; and we write $X\otimes Y\otimes Z$ both instead of $(X\otimes Y)\otimes Z$ and $X\otimes(Y\otimes Z)$, for any objects $X, Y, Z$.

**Definition 2.5.** A *monoidal structure* on a functor $f$ from a monoidal category $\mathsf{A}$ to a monoidal category $\mathsf{A}'$ consists of

- a morphism $f^0 : I' \to fI$ (called the *nullary part*)
- a natural transformation $f^2 : f - \otimes' f - \to f(- \otimes -)$ (called the *binary part*)

such that the following diagrams commute for any objects $X, Y, Z$ of $\mathsf{A}$.

$$\begin{tikzcd}
fX \otimes' fY \otimes' fZ \arrow[r, "f^2_{X,Y} \otimes' 1"] \arrow[d, "1 \otimes' f^2_{Y,Z}"'] & f(X \otimes Y) \otimes' fZ \arrow[d, "f^2_{X \otimes Y, Z}"] \\
fX \otimes' f(Y \otimes Z) \arrow[r, "f^2_{X, Y \otimes Z}"'] & f(X \otimes Y \otimes Z)
\end{tikzcd}
\qquad
\begin{tikzcd}
fX \arrow[r, "f^0 \otimes' 1"] \arrow[d, "1 \otimes' f^0"'] \arrow[dr, equal] & fI \otimes' fX \arrow[d, "f^2_{I,X}"] \\
fX \otimes' fI \arrow[r, "f^2_{X,I}"'] & fX
\end{tikzcd}$$

An *opmonoidal structure* on $f$ consists of

- a morphism $f^0 : fI \to I'$ (called the *nullary part*)
- a natural transformation $f^2 : f(- \otimes -) \to f - \otimes' f -$ (called the *binary part*)

such that the same diagrams with reversed arrows commute.

An (op)monoidal structure is *strong* if the morphism $f^0$ and the natural transformation $f^2$ are invertible.

An (op)monoidal structure is *strict* if $f^0$ is the identity morphism and $f^2$ is the identity natural transformation. This means the equality of objects $fI = I'$ and $f(X \otimes Y) = fX \otimes fY$; and the conditions $\alpha'_{fX,fY,fZ} = f\alpha_{X,Y,Z}$, $\lambda'_{fX} = f\lambda_X$ and $\rho'_{fX} = f\rho_X$ on the (un-denoted) associativity and unit constraints, for all objects $X, Y, Z$ of $\mathsf{A}$.

**Examples 2.6.**

(1) Identity functors on monoidal categories are strict monoidal.
(2) The 'linear span' functor in part (2) of Examples 1.4 is strong monoidal.
(3) Take an algebra $A$ over a field $k$. The forgetful functor $u$ (see part (3) of Examples 1.4) from the monoidal category category of $A$-bimodules in part (3) of Examples 2.2 to the monoidal category of vector spaces in part (2) of Examples 2.2, admits the following monoidal structure. The nullary part is the linear map $k \to A$ sending the multiplicative unit 1 of $k$ (i.e. the "number" 1) to the unit element of the algebra $A$, and the binary part $uV \otimes_k uW \to u(V \otimes_A W)$ is given by the canonical projection, for any $A$-bimodules $V$ and $W$. This monoidal structure is not even strong.

**Exercise 2.7.** Show that the composite of (op)monoidal functors is (op)monoidal.

*Hint.* Take monoidal functors $\mathsf{A} \xrightarrow{f} \mathsf{B} \xrightarrow{g} \mathsf{C}$. Construct a monoidal structure on their composite with

nullary part $I \xrightarrow{g^0} gI \xrightarrow{gf^0} gfI$ and

binary part $gfX \otimes gfY \xrightarrow{g^2_{fX,fY}} g(fX \otimes fY) \xrightarrow{gf^2_{X,Y}} gf(X \otimes Y)$.

**Exercise 2.8.** Prove that in an adjunction $l \dashv r$ between monoidal categories, there is a bijective correspondence between the monoidal structures on $r$ and the opmonoidal structures on $l$.

*Hint.* Take an adjunction $l \dashv r : \mathsf{A} \to \mathsf{A}'$ between monoidal categories, with unit $\eta$ and counit $\varepsilon$. In terms of a monoidal structure $(r^0, r^2)$ on $r$, construct an opmonoidal structure on $l$ with nullary part $lI' \xrightarrow{lr^0} lrI \xrightarrow{\varepsilon_I} I$ and binary part $l(X \otimes' Y) \xrightarrow{l(\eta_X \otimes' \eta_Y)} l(rlX \otimes' rlY) \xrightarrow{lr^2_{lX,lY}} lr(lX \otimes lY) \xrightarrow{\varepsilon_{lX \otimes lY}} lX \otimes lY$ for all objects $X,Y$ of $\mathsf{A}'$. Construct the inverse map symmetrically.

**Definition 2.9.** A natural transformation $\varphi : f \to f'$ between monoidal functors is said to be *monoidal* if the following diagrams commute.

$$\begin{tikzcd}
fX \otimes' fY \arrow[r, "f^2_{X,Y}"] \arrow[d, "\varphi_X \otimes' \varphi_Y"'] & f(X \otimes Y) \arrow[d, "\varphi_{X \otimes Y}"] \\
f'X \otimes' f'Y \arrow[r, "f'^2_{X,Y}"'] & f'(X \otimes Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
I' \arrow[r, "f^0"] \arrow[d, equal] & fI \arrow[d, "\varphi_I"] \\
I' \arrow[r, "f'^0"'] & f'I
\end{tikzcd}$$

A natural transformation $\varphi : f \to f'$ between opmonoidal functors is said to be *opmonoidal* if the same diagrams with reversed horizontal arrows commute.

**Example 2.10.** Identity natural transformations of (op)monoidal functors are (op)monoidal.

**Exercise 2.11.** Show that both the composite and the Godement product of (op)monoidal natural transformations is (op)monoidal (with respect to the (op)monoidal structure of the composite functors in Exercise 2.7).

**Definition 2.12.** An *opmonoidal monad* on a monoidal category $\mathsf{A}$ consists of

- a monad $t$ on the category $\mathsf{A}$ (with multiplication $\mu$ and unit $\eta$)
- an opmonoidal structure $(t^0 : tI \to I, t^2 : t(- \otimes -) \to t - \otimes t-)$ on the functor $t$

such that $\mu$ and $\eta$ are opmonoidal natural transformations; that is, the following diagrams commute. (The morphisms in the top row of the first diagram in each row are computed as in Exercise 2.7.)

$$\begin{tikzcd}
t^2(X \otimes Y) \arrow[r, "tt^2_{X,Y}"] \arrow[d, "\mu_{X \otimes Y}"'] & t(tX \otimes tY) \arrow[r, "t^2_{tX,tY}"] & t^2X \otimes t^2Y \arrow[d, "\mu_X \otimes \mu_Y"] \\
t(X \otimes Y) \arrow[rr, "t^2_{X,Y}"'] & & tX \otimes tY
\end{tikzcd}
\qquad
\begin{tikzcd}
X \otimes Y \arrow[r, equal] \arrow[d, "\eta_{X \otimes Y}"'] & X \otimes Y \arrow[d, "\eta_X \otimes \eta_Y"] \\
t(X \otimes Y) \arrow[r, "t^2_{X,Y}"'] & tX \otimes tY
\end{tikzcd}$$

$$\begin{tikzcd}
t^2I \arrow[r, "tt^0"] \arrow[d, "\mu_I"'] & tI \arrow[r, "t^0"] & I \arrow[d, equal] \\
tI \arrow[rr, "t^0"'] & & I
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, equal] \arrow[d, "\eta_I"'] & I \arrow[d, equal] \\
tI \arrow[r, "t^0"'] & I
\end{tikzcd}$$

**Theorem 2.13.** [23] [21] *Consider a monoidal category $\mathsf{A}$ and a monad $t$ on the category $\mathsf{A}$ (with multiplication $\mu$ and unit $\eta$). There is a bijective correspondence between the following data.*

(i) *Monoidal structures on the Eilenberg–Moore category $\mathsf{A}^t$ such that the forgetful functor $u^t : \mathsf{A}^t \to \mathsf{A}$ is strict monoidal.*

(ii) *Liftings of the monoidal structure of* $\mathsf{A}$ *to* $\mathsf{A}^t$.
(iii) *Opmonoidal monad structures on* $t$.

*For this reason, opmonoidal monads are also called* bimonads.

*Proof.* Recall from Definition 2.5 that the structure in part (i) means, in fact, a monoidal structure on $\mathsf{A}^t$ such that the object part of the monoidal unit $\overline{I}$ is the monoidal unit $I$ of $\mathsf{A}$, the object part of the monoidal product $(V,v)\overline{\otimes}(W,w)$ of Eilenberg–Moore $t$-algebras $(V,v)$ and $(W,w)$ is the monoidal product $V\otimes W$ in $\mathsf{A}$ and the associativity and unit constraints satisfy

$$u^t\overline{\alpha}_{(V,v),(W,w),(Z,z)} = \alpha_{V,W,Z}, \qquad u^t\overline{\lambda}_{(V,v)} = \lambda_V, \qquad u^t\overline{\rho}_{(V,v)} = \rho_V$$

for all Eilenberg–Moore $t$-algebras $(V,v),(W,w)$ and $(Z,z)$. In other words, regarding $I$ as a functor from the singleton category $\mathbb{1}$ to $\mathsf{A}$, the structure in part (i) consists of

- a lifting of the functor $I:\mathbb{1}\to\mathsf{A}$ to a functor $\overline{I}:\mathbb{1}\to\mathsf{A}^t$
- a lifting of the functor $\otimes:\mathsf{A}\times\mathsf{A}\to\mathsf{A}$ to a functor $\overline{\otimes}:\mathsf{A}^t\times\mathsf{A}^t\to\mathsf{A}^t$
- a lifting of the natural transformation $\alpha:\otimes\circ(\otimes\times 1)\to\otimes\circ(1\times\otimes)$ to a natural transformation $\overline{\alpha}:\overline{\otimes}\circ(\overline{\otimes}\times 1)\to\overline{\otimes}\circ(1\times\overline{\otimes})$
- a lifting of the natural transformation $\lambda:\otimes\circ(I\times 1)\to 1$ to $\overline{\lambda}:\overline{\otimes}\circ(\overline{I}\times 1)\to 1$
- a lifting of the natural transformation $\rho:\otimes\circ(1\times I)\to 1$ to $\overline{\rho}:\overline{\otimes}\circ(1\times\overline{I})\to 1$.

That is, a lifting of the monoidal structure as in (ii).

By Theorem & Definition 1.19, the first two items in the above list correspond bijectively to monad morphisms $t^0:tI\to I$ and $t^2:t(-\otimes-)\to t-\otimes t-$. The diagrams expressing that $t^0$ and $t^2$ are monad morphisms, are identical to the diagrams presented in Definition 2.12.

By Theorem & Definition 1.22, the data in the last three items of the above list exist if and only if $t^0$ and $t^2$ satisfy certain compatibility conditions with the associativity and the unit constraints. Spelling out these compatibility conditions, we get precisely the diagrams in Definition 2.5 defining an opmonoidal structure $(t^0,t^2)$ on $t$.

That is, $(t^0,t^2)$ equip $t$ with the structure of an opmonoidal monad as in part (iii). $\square$

**Definition 2.14.** A monoidal category $\mathsf{A}$ said to be is *right closed* if for any object $X$, the functor $-\otimes X$ possesses a right adjoint. The right adjoint is called the *internal hom* functor and it is denoted by $[X,-]$. The unit and the counit of the adjunction $-\otimes X\dashv[X,-]$ will be denoted by $\eta^X$ and $\varepsilon^X$, respectively.

**Examples 2.15.**

(1) The monoidal category of sets is right closed, with internal hom $[X,Y]$ the set of maps $X\to Y$.
(2) The monoidal category of vector spaces is right closed, with internal hom $[X,Y]$ the vector space of linear maps $X\to Y$.
(3) The monoidal category of bimodules over an arbitrary algebra $A$ is right closed, with internal hom $[X,Y]$ the $A$-bimodule of right $A$-module maps $X\to Y$ (the left and right $A$-actions on a right $A$-module map $h:X\to Y$ are given by $(a\cdot h\cdot a')x := a\cdot h(a'\cdot x)$).

**Definition 2.16.** A strict monoidal functor $u:\mathsf{A}'\to\mathsf{A}$ between right closed monoidal categories *strictly preserves* the right closed structure if each component of the natural transformation

$$u[X,Y]' \xrightarrow{\eta^{uX}_{u[X,Y]'}} [uX,u[X,Y]'\otimes uX] = [uX,u([X,Y]'\otimes' X)] \xrightarrow{[uX,u\varepsilon'^X_Y]} [uX,uY]$$

is the identity morphism.

**Lemma 2.17.** *For a strictly monoidal functor $u : \mathsf{A}' \to \mathsf{A}$ the following assertions are equivalent.*

(i) *It strictly preserves the right closed structure.*

(ii) *The functor equality $u[-,-]' = [u-,u-]$ holds and the following diagram commutes for any objects $X,Y$ of $\mathsf{A}'$.*

$$\begin{tikzcd}
uY \arrow[r, "\eta^{uX}_{uY}"] \arrow[d, "u\eta'^{X}_{Y}"'] & {[uX, uY \otimes uX]} \arrow[d, equal] \\
u[X, Y \otimes' X]' \arrow[r, equal] & {[uX, u(Y \otimes' X)]}
\end{tikzcd} \tag{2.2}$$

(iii) *The functor equality $u[-,-]' = [u-,u-]$ holds and the following diagram commutes for any objects $X,Y$ of $\mathsf{A}'$.*

$$\begin{tikzcd}
u[X,Y]' \otimes uX \arrow[r, equal] \arrow[d, equal] & u([X,Y]' \otimes' X) \arrow[d, "u\varepsilon'^{X}_{Y}"] \\
{[uX, uY] \otimes uX} \arrow[r, "\varepsilon^{uX}_{uY}"'] & uY
\end{tikzcd} \tag{2.3}$$

*Proof.* This follows by the triangle conditions. Assertion (i) implies (ii) by the commutativity of

$$\begin{tikzcd}
uY \arrow[r, "\eta^{uX}_{uY}"] \arrow[dd, "u\eta'^{X}_{Y}"'] & {[uX, uY \otimes uX]} \arrow[rr, equal] \arrow[d, "{[uX, u\eta'^{X}_{Y} \otimes 1]}"] & & {[uX, u(Y \otimes' X)]} \arrow[dl, "{[uX, u(\eta'^{X}_{Y} \otimes' 1)]}"'] \arrow[dd, equal] \\
& {[uX, u[X, Y \otimes' X]' \otimes uX]} \arrow[r, equal] & {[uX, u([X, Y \otimes' X]' \otimes' X)]} \arrow[dr, "{[uX, u\varepsilon'^{X}_{Y \otimes' X}]}"'] & \\
u[X, Y \otimes' X]' \arrow[ur, "\eta^{uX}_{u[X, Y \otimes' X]'}"] \arrow[rrr, equal] & & & {[uX, u(Y \otimes' X)]}.
\end{tikzcd}$$

Conversely, (ii) implies (i) by the commutativity of

$$\begin{tikzcd}
u[X,Y]' \arrow[rr, equal] \arrow[dr, "u\eta'^{X}_{[X,Y]'}"] \arrow[dd, "\eta^{uX}_{u[X,Y]'}"'] & & u[X,Y]' \arrow[dd, equal] \\
& u[X, [X,Y]' \otimes' X]' \arrow[ur, "{u[X, \varepsilon'^{X}_{Y}]'}"] \arrow[d, equal] & \\
{[uX, u[X,Y]' \otimes uX]} \arrow[r, equal] \arrow[ur, phantom, "(2.2)"] & {[uX, u([X,Y]' \otimes' X)]} \arrow[r, "{[uX, u\varepsilon'^{X}_{Y}]}"'] & {[uX, uY]}.
\end{tikzcd}$$

The equivalence (i)⇔(iii) follows symmetrically. □

**Theorem & Definition 2.18.** [15, Theorem 3.6] *On an opmonoidal monad (a.k.a. bimonad) $t$ on a right closed monoidal category $\mathsf{A}$ the following assertions are equivalent.*

(i) *$\mathsf{A}^t$ is right closed monoidal and the forgetful functor $u^t$ strictly preserves the right closed structure.*

(ii) *The right closed structure of the monoidal category* $\mathsf{A}$ *lifts to* $\mathsf{A}^t$.

(iii) *The natural transformation*

$$t(-\otimes t-) \xrightarrow{t^2} t - \otimes t^2 - \xrightarrow{1\otimes\mu_-} t - \otimes t -$$

*(between functors* $\mathsf{A}\times\mathsf{A}\to\mathsf{A}$*) is invertible.*

*In this situation $t$ is termed a* Hopf monad.

*Proof.* Since the forgetful functor $u^t : \mathsf{A}^t \to \mathsf{A}$ is strict monoidal, the functor $-\otimes(V,v) : \mathsf{A}^t \to \mathsf{A}^t$ is a lifting of $-\otimes V : \mathsf{A} \to \mathsf{A}$ for any Eilenberg–Moore $t$-algebra $(V,v)$, along the monad morphism

$$\beta_{X,(V,v)} := \ t(X\otimes V) \xrightarrow{t^2_{X,V}} tX\otimes tV \xrightarrow{1\otimes v} tX\otimes V \tag{2.4}$$

for any object $X$ of $\mathsf{A}$.

By Lemma 2.17, assertion (i) is equivalent to the lifting of the adjunction $-\otimes V \dashv [V,-] : \mathsf{A}\to\mathsf{A}$ to some adjunction $-\otimes(V,v) \dashv [(V,v),-]' : \mathsf{A}^t \to \mathsf{A}^t$ for any Eilenberg–Moore $t$-algebra $(V,v)$. This proves (i)⇔(ii).

By Theorem & Definition 1.24, (ii) is equivalent to the invertibility of $\beta_{X,(V,v)}$ of (2.4) for all objects $X$ of $\mathsf{A}$ and all objects $(V,v)$ of $\mathsf{A}^t$.

If $\beta_{X,(V,v)}$ is invertible for all objects $X$ of $\mathsf{A}$ and all objects $(V,v)$ of $\mathsf{A}^t$, then in particular $\beta_{X,f^tY}$ is invertible for all objects $X,Y$ of $\mathsf{A}$ (where $f^t$ is the left adjoint of $u^t$ from Paragraph 1.18). This proves (ii)⇒(iii).

Conversely, assume that (iii) holds; that is, $\beta_{X,f^tY}$ is invertible for all objects $X,Y$ of $\mathsf{A}$. Then since via the $t$-module epimorphism $v : f^tV = (tV,\mu_V) \to (V,v)$ any Eilenberg–Moore $t$-algebra $(V,v)$ is the quotient of a free one $f^tV$, also $\beta_{X,(V,v)}$ possesses the inverse

$$tX\otimes V \xrightarrow{1\otimes\eta^t_V} tX\otimes tV \xrightarrow{\beta^{-1}_{X,f^tV}} t(X\otimes tV) \xrightarrow{t(1\otimes v)} t(X\otimes V).$$

It is seen to be the two-sided inverse of $\beta_{X,(V,v)}$ by the commutativity of the following diagrams. (Here we use the naturality of $\eta^t$ and $\beta$, and the unitality and associativity of $\mu^t$ and $v$.)

$$\begin{tikzcd}
tX\otimes V \arrow[r, "1\otimes\eta^t_V"] \arrow[dddrrr, equals, bend right] & tX\otimes tV \arrow[r, "\beta^{-1}_{X,f^tV}"] \arrow[dr, equals] & t(X\otimes tV) \arrow[r, "t(1\otimes v)"] \arrow[d, "\beta_{X,f^tV}"] & t(X\otimes V) \arrow[ddd, "\beta_{X,(V,v)}"] \\
 & & tX\otimes tV \arrow[ddr, "1\otimes v"] & \\
 & & & \\
 & & & tX\otimes V
\end{tikzcd}$$

$$
\begin{tikzcd}
t(X \otimes v) \arrow[rrr, equal] \arrow[dr, "t(1\otimes\eta^t_V)"] \arrow[dddddd, equal] & & & t(X \otimes V) \arrow[dd, "\beta_{X,(V,v)}"] \\
& t(X \otimes tV) \arrow[urr, "t(1\otimes v)"] \arrow[dd, "\beta_{X,f^tV}"] \arrow[dddd, equal, bend right] & & \\
& & tX \otimes tV \arrow[r, "1\otimes v"] \arrow[dl, equal, bend right] \arrow[d, "1\otimes\eta^t_{tV}"] & tX \otimes V \arrow[d, "1\otimes\eta^t_V"] \\
& tX \otimes tV \arrow[dd, "\beta^{-1}_{X,f^tV}"] & tX \otimes t^2V \arrow[l, "1\otimes\mu^t_V"'] \arrow[r, "1\otimes tv"] \arrow[d, "\beta^{-1}_{X,f^ttV}"] & tX \otimes tV \arrow[d, "\beta^{-1}_{X,f^tV}"] \\
& & t(X \otimes t^2V) \arrow[r, "t(1\otimes tv)"'] \arrow[dl, "t(1\otimes\mu^t_V)"] & t(X \otimes tV) \arrow[dd, "t(1\otimes v)"] \\
& t(X \otimes tV) \arrow[drr, "t(1\otimes v)"] & & \\
t(X \otimes V) \arrow[rrr, equal] \arrow[ur, "t(1\otimes\eta^t_V)"'] & & & t(X \otimes V)
\end{tikzcd}
$$

□

Let us stress that for general Hopf monoids $t$ there needs to be no antipode of the type $t \to t$.

## 3. Lecture: (Hopf) bialgebras and (Hopf) bialgebroids

In the third lecture we first see how the classical structure of (Hopf) bialgebra (over a field) fits the framework developed in the second lecture. The next example to be discussed is that of a *(Hopf) bialgebroid* (over an arbitrary base algebra). The most important references are [32, Theorems 4.4 & 4.5] and [26, Theorem 5.1].

By part (2) of Examples 2.2, the category $\mathsf{vec}$ of vector spaces (over some field $k$) possesses a monoidal structure. Take a vector space $A$; recall from part (4) of Examples 1.4 the induced functor $A \otimes - : \mathsf{vec} \to \mathsf{vec}$. We begin with the investigation of the opmonoidal monad structures on it.

**Proposition 3.1.** *For any vector space $A$ over a field $k$, there is a bijective correspondence between the following structures.*

(i) *monad structures on the functor $A \otimes - : \mathsf{vec} \to \mathsf{vec}$ (see part (4) of Examples 1.4)*
(ii) *algebra structures on the vector space $A$*

*Furthermore, in this setting the Eilenberg–Moore algebra of the monad $A \otimes - : \mathsf{vec} \to \mathsf{vec}$ in part (i) is the category $\mathsf{mod}(A)$ of modules over the algebra $A$ in part (ii).*

*Proof.* An algebra $A$ induces a monad $A \otimes - : \mathsf{vec} \to \mathsf{vec}$ as in part (3) of Examples 1.15.

Conversely, a monad structure $(\eta, \mu)$ on the functor $A \otimes - : \mathsf{vec} \to \mathsf{vec}$ determines an algebra structure on the vector space $A$ with unit map $\eta_k : k \to A$ and multiplication map $\mu_k : A \otimes A \to A$; associativity and unitality are immediate.

Starting with an algebra $A$ and iterating these constructions we evidently re-obtain the same algebra $A$. In order to see that starting with a monad $(A \otimes -, \eta, \mu)$ and iterating these constructions in the opposite order we also re-obtain the same monad, we use the same type of reasoning as in Exercise 1.7. Namely, observe that any element $v$ of an arbitrary vector space $V$ induces a linear map $k \to V$ sending $1 \in k$ to $v$. Then it follows by the naturality diagrams

$$\begin{tikzcd} k \arrow[r, "\eta_k"] \arrow[d, "v"'] & A \arrow[d, "1\otimes v"] \\ V \arrow[r, "\eta_V"'] & A\otimes V \end{tikzcd} \qquad \begin{tikzcd} A\otimes A \arrow[r, "\mu_k"] \arrow[d, "1\otimes 1\otimes v"'] & A \arrow[d, "1\otimes v"] \\ A\otimes A\otimes V \arrow[r, "\mu_V"'] & A\otimes V \end{tikzcd}$$

that $\eta_V(v) = \eta_k(1)\otimes v$ and $\mu_V(a'\otimes a\otimes v) = \mu_k(a'\otimes a)\otimes v$.

The final claim can be found in part (2) of Examples 1.17. □

**Proposition 3.2.** *(1) For any vector space $C$ over a field $k$, there is a bijective correspondence between the following structures.*

- (i) *opmonoidal structures on the functor $C\otimes - : \mathsf{vec} \to \mathsf{vec}$ (see part (4) of Examples 1.4)*
- (ii) *coalgebra structures on the vector space $C$*

*(2) For two coalgebras $C$ and $C'$ and a linear map $f : C \to C'$, the following are equivalent.*

- (i) *For all vector spaces $V$, the maps $f\otimes 1 : C\otimes V \to C'\otimes V$ are the components of an opmonoidal natural transformation $C\otimes - \to C'\otimes -$ between the opmonoidal functors in part (1).*
- (ii) *$f$ is a homomorphism of coalgebras (that is, for all elements $c$ of $C$, $e'(f(c)) = e(c)$ and $f(c)_1\otimes f(c)_2 = f(c_1)\otimes f(c_2)$, where a Sweedler-Heyneman type implicit summation index notation is used.)*

*Proof.* (1) If $C$ has a comultiplication $c \mapsto c_1\otimes c_2$ with counit $e$, then an opmonoidal structure on the functor $t := C\otimes - : \mathsf{vec} \to \mathsf{vec}$ is provided by the nullary part $e : C \to k$ and the binary part

$$t^2_{V,W} : C\otimes V\otimes W \to C\otimes V\otimes C\otimes W, \qquad c\otimes v\otimes w \mapsto c_1\otimes v\otimes c_2\otimes w$$

for any vector spaces $V$ and $W$. Commutativity of the diagrams of Definition 2.9 is immediate from the coassociativity and counitality of the coalgebra $C$:

$$\begin{tikzcd} c\otimes v\otimes w\otimes z \arrow[rr, mapsto, "t^2_{V\otimes W,Z}"] \arrow[d, mapsto, "t^2_{V,W\otimes Z}"'] & & c_1\otimes v\otimes w\otimes c_2\otimes z \arrow[d, mapsto, "t^2_{V,W}\otimes 1"] \\ c_1\otimes v\otimes c_2\otimes w\otimes z \arrow[r, mapsto, "1\otimes t^2_{W,Z}"'] & c_1\otimes v\otimes c_{21}\otimes w\otimes c_{22}\otimes z \arrow[r, equal] & c_{11}\otimes v\otimes c_{12}\otimes w\otimes c_2\otimes z \end{tikzcd}$$

$$\begin{tikzcd} c\otimes v \arrow[rr, mapsto, "t^2_{k,V}"] \arrow[d, mapsto, "t^2_{V,k}"'] \arrow[drr, equal] & & c_1\otimes c_2\otimes v \arrow[d, mapsto, "e\otimes 1\otimes 1"] \\ c_1\otimes v\otimes c_2 \arrow[r, mapsto, "1\otimes 1\otimes e"'] & c_1 e(c_2)\otimes v \arrow[r, equal] & e(c_1)c_2\otimes v \end{tikzcd}$$

commute for any element $c$ of $C$, for any vector spaces $V,W,Z$ and elements $v\in V$, $w\in W$ and $z\in Z$.

Conversely, an opmonoidal structure $(t^0,t^2)$ on the functor $t := C\otimes - : \mathsf{vec} \to \mathsf{vec}$ determines a coassociative comultiplication $t^2_{k,k} : C \to C\otimes C$ with counit $t^0$.

Starting with a coalgebra $C$ and iterating these constructions we evidently re-obtain the same coalgebra $C$. In order to see that starting with an opmonoidal functor $(t = C \otimes -, t^0, t^2)$ and iterating these constructions in the opposite order we also re-obtain the same opmonoidal structure $(t^0, t^2)$, take the linear maps $v : k \to V$ and $w : k \to W$ induced by arbitrary elements $v$ and $w$ of respective vector spaces $V$ and $W$. Then it follows by the naturality of $t^2$ that the following diagram commutes.

$$\begin{tikzcd}
C \arrow[r, "t^2_{k,k}"] \arrow[d, "1\otimes v\otimes w"'] & C \otimes C \arrow[d, "1\otimes v\otimes 1\otimes w"] \\
C \otimes V \otimes W \arrow[r, "t^2_{V,W}"'] & C \otimes V \otimes C \otimes W
\end{tikzcd}$$

This completes the proof of part (1).

(2) In light of part (1), opmonoidality of the natural transformation with components $f \otimes 1 : C \otimes V \to C' \otimes V$ translates to

$$e'(f(c)) = e(c) \quad \text{and} \quad f(c)_1 \otimes v \otimes f(c)_2 \otimes w = f(c_1) \otimes v \otimes f(c_2) \otimes w$$

for any vector spaces $V$ and $W$, $v \in V$, $w \in W$ and $c \in C$. This is clearly equivalent to $f$ being a coalgebra map. $\square$

**Theorem & Example 3.3.** *For any vector space $T$ (over some field $k$) there is a bijective correspondence between the following structures.*

(i) *bimonad structures on the functor $T \otimes - : \mathsf{vec} \to \mathsf{vec}$*
(ii) *bialgebra structures on the vector space $T$*

*Moreover, $T \otimes - : \mathsf{vec} \to \mathsf{vec}$ is a Hopf monad if and only if $T$ is a Hopf algebra.*

*Proof.* Proposition 3.1 provides us with a bijection between the monad structures on $t := T \otimes -$ and the algebra structures on $T$. Part (1) of Proposition 3.2 provides us with a bijection between the opmonoidal structures on $t$ and the coalgebra structures on $T$. By part (2) of Proposition 3.2 the multiplication and the unit of the monad $t$ are opmonoidal natural transformations if and only if the multiplication and the unit of the algebra $T$ are coalgebra homomorphisms. That is, if and only if the algebra and coalgebra structures of $T$ combine to a bialgebra.

Now let $T$ be a bialgebra, equivalently, let $t = T \otimes -$ be a bimonad. Then $t$ is a Hopf monad if and only if for any vector spaces $V$ and $W$, the map

$$\beta_{V,T\otimes W} : T \otimes V \otimes T \otimes W \to T \otimes V \otimes T \otimes W, \qquad c \otimes v \otimes c' \otimes w \mapsto c_1 \otimes v \otimes c_2 c' \otimes w$$

is invertible. This is clearly equivalent to the invertibility of the map

$$\beta_{k,T} : T \otimes T \to T \otimes T, \qquad c \otimes c' \mapsto c_1 \otimes c_2 c'$$

rendering commutative the following diagrams.

$$\begin{tikzcd}
T\otimes T\otimes T \arrow[r, "\beta_{k,T}\otimes 1"] \arrow[d, "1\otimes\mu_k"] & T\otimes T\otimes T \arrow[d, "1\otimes\mu_k"'] \\
T\otimes T \arrow[r, "\beta_{k,T}"'] & T\otimes T
\end{tikzcd}
\quad
\begin{tikzcd}
T\otimes T \arrow[r, "\beta_{k,T}"] \arrow[d, "t^2_{k,k}\otimes 1"] & T\otimes T \arrow[d, "t^2_{k,k}\otimes 1"'] \\
T\otimes T\otimes T \arrow[r, "1\otimes\beta_{k,T}"'] & T\otimes T\otimes T
\end{tikzcd}
\quad
\begin{tikzcd}
T\otimes T \arrow[r, "\beta_{k,T}"] \arrow[dr, "\mu_k"'] & T\otimes T \arrow[d, "t^0\otimes 1"] \\
 & T
\end{tikzcd}
\quad
\begin{tikzcd}
T \arrow[r, "1\otimes\eta_k"] \arrow[dr, "t^2_{k,k}"'] & T\otimes T \arrow[d, "\beta_{k,T}"] \\
 & T\otimes T
\end{tikzcd}
\tag{3.1}$$

If $\beta_{k,T}$ is invertible, then $T$ is a Hopf algebra with the antipode

$$T \xrightarrow{1\otimes\eta_k} T\otimes T \xrightarrow{\beta_{k,T}^{-1}} T\otimes T \xrightarrow{t^0\otimes 1} T.$$

Indeed, using the commutativity of the diagrams of (3.1) also the following diagrams commute.

$$\begin{tikzcd}
T\otimes T \arrow[r, "1\otimes\eta_k\otimes 1"] \arrow[dr, equal] & T\otimes T\otimes T \arrow[r, "\beta_{k,T}^{-1}\otimes 1"] \arrow[d, "1\otimes\mu_k"] & T\otimes T\otimes T \arrow[r, "t^0\otimes 1\otimes 1"] \arrow[d, "1\otimes\mu_k"] & T\otimes T \arrow[dd, "\mu_k"] \\
& T\otimes T \arrow[r, "\beta_{k,T}^{-1}"] & T\otimes T \arrow[dr, "t^0\otimes 1"] & \\
T \arrow[uu, "t^2_{k,k}"] \arrow[urr, "1\otimes\eta_k"] \arrow[r, "t^0"'] & k \arrow[rr, "\eta_k"'] & & T
\end{tikzcd}$$

$$\begin{tikzcd}
T\otimes T \arrow[r, "1\otimes 1\otimes\eta_k"] & T\otimes T\otimes T \arrow[r, "1\otimes\beta_{k,T}^{-1}"] & T\otimes T\otimes T \arrow[r, "1\otimes t^0\otimes 1"] & T\otimes T \arrow[dd, "\mu_k"] \arrow[dl, equal] \\
& T\otimes T \arrow[u, "t^2_{k,k}\otimes 1"] \arrow[r, "\beta_{k,T}^{-1}"] \arrow[drr, "t^0\otimes 1"] & T\otimes T\otimes T \arrow[u, "t^2_{k,k}\otimes 1\otimes 1"'] & \\
T \arrow[uu, "t^2_{k,k}"] \arrow[ur, "1\otimes\eta_k"] \arrow[r, "t^0"'] & k \arrow[rr, "\eta_k"'] & & T
\end{tikzcd}$$

Conversely, if $T$ is a Hopf algebra with the antipode $\sigma$, then $\beta_{k,T}$ has the inverse

$$\beta_{k,T}^{-1} : T\otimes T \to T\otimes T, \qquad c\otimes c' \mapsto c_1\otimes\sigma(c_2)c'.$$

Indeed,

$$\begin{aligned}
\beta_{k,T}^{-1}\circ\beta_{k,T}(c\otimes c') &= \beta_{k,T}^{-1}(c_1\otimes c_2c') = c_{11}\otimes\sigma(c_{12})(c_2c') \\
&= c_1\otimes(\sigma(c_{21})c_{22})c' = c_1t^0(c_2)\otimes c' = c\otimes c' \quad \text{and} \\
\beta_{k,T}\circ\beta_{k,T}^{-1}(c\otimes c') &= \beta_{k,T}(c_1\otimes\sigma(c_2)c') = c_{11}\otimes c_{12}(\sigma(c_2)c') \\
&= c_1\otimes(c_{21}\sigma(c_{22}))c' = c_1t^0(c_2)\otimes c' = c\otimes c'.
\end{aligned}$$

□

Our next aim is to derive the axioms of a Takeuchi bialgebroid [35] and of a Schauenburg Hopf algebroid [26] in a similar manner, by lifting the monoidal and the right closed structures of a bimodule category to a suitable Eilenberg-Moore category. That is, as bimonad and Hopf monad structures on a suitable functor. We shall follow similar steps as before: we study separately the possible monad structures, the opmonoidal structures, and finally their compatibility.

We start with describing the underlying functor.

**3.4. Modules bimodules and more.** For any algebra $B$ over a field, we may consider the *opposite algebra* $B^{\mathsf{op}}$. It lives on the same vector space $B$ but it has the opposite multiplication $b\otimes b'\mapsto b'b$ (where juxtaposition stands for the multiplication of $B$). Clearly, $(B^{\mathsf{op}})^{\mathsf{op}}$ is the algebra $B$.

Consider now the *enveloping algebra* $B^{\mathrm{e}} := B \otimes B^{\mathrm{op}}$ (with the factorwise multiplication). Any $B$-bimodule $V$ can be regarded as a left module over $B^{\mathrm{e}}$ via the action

$$B^{\mathrm{e}} \otimes V \to V, \qquad b \otimes b' \otimes v \mapsto b \cdot v \cdot b'.$$

Together with the identity map on the morphisms this defines an isomorphism between the category $\mathsf{mod}(B^{\mathrm{e}})$ of left $B^{\mathrm{e}}$-modules and the category $\mathsf{bim}(B)$ of $B$-bimodules.

Take next a $B^{\mathrm{e}}$-bimodule $V$. As in part (4) of Examples 1.4, it defines a functor $V \otimes_{B^{\mathrm{e}}} - : \mathsf{mod}(B^{\mathrm{e}}) \to \mathsf{mod}(B^{\mathrm{e}})$. Composing it on both ends with the isomorphism $\mathsf{mod}(B^{\mathrm{e}}) \cong \mathsf{bim}(B)$ of the previous paragraph, we obtain a functor $\mathsf{bim}(B) \to \mathsf{bim}(B)$. It will be denoted by $V \boxtimes -$. Remember that it sends a $B$-bimodule $W$ to the quotient of the vector space $V \otimes W$ with respect to the subspace

$$\{v \cdot (b \otimes b') \otimes w - v \otimes b \cdot w \cdot b' | v \in V,\ w \in W,\ b \otimes b' \in B^{\mathrm{e}}\}$$

with the actions

$$B \otimes (V \boxtimes W) \otimes B \to V \boxtimes W, \qquad b \otimes (v \boxtimes w) \otimes b' \mapsto (b \otimes b') \cdot v \ \boxtimes\ w.$$

In particular, considering $B^{\mathrm{e}}$ as a $B$-bimodule $_{|}B^{\mathrm{e}}$ via the actions

$$B \otimes B^{\mathrm{e}} \otimes B \to B^{\mathrm{e}}, \qquad b \otimes (p \otimes q) \otimes b' \mapsto bp \otimes qb', \tag{3.2}$$

$V \boxtimes {_{|}B^{\mathrm{e}}}$ is isomorphic to the $B$-bimodule to be denoted by $_{|}V$, defined by the $B$-actions on $V$ in

$$B \otimes V \otimes B \to V, \qquad b \otimes v \otimes b' \mapsto (b \otimes b') \cdot v. \tag{3.3}$$

**Proposition 3.5.** *For any algebra $B$ and any $B^{\mathrm{e}}$-bimodule $A$, there is a bijective correspondence between the following structures.*

(i) *monad structures on the functor $A \boxtimes - : \mathsf{bim}(B) \to \mathsf{bim}(B)$ (see Paragraph 3.4)*
(ii) *algebra structures on $A$ together with an algebra homomorphism $B^{\mathrm{e}} \to A$ such that the $B^{\mathrm{e}}$-actions on $A$ are induced by this homomorphism.*

*The structure in part (ii) is called a $B^{\mathrm{e}}$-*ring *structure on $A$.*

*Furthermore, in this setting the Eilenberg–Moore algebra of the monad $A \boxtimes - : \mathsf{bim}(B) \to \mathsf{bim}(B)$ in part (i) is isomorphic to the category $\mathsf{mod}(A)$ of modules over the algebra $A$ in part (ii).*

*Proof.* The structure in part (i) consists of a natural transformation $\eta$ with components $\eta_W : W \to A \boxtimes W$, and a natural transformation $\mu$ with components $\mu_W : A \boxtimes (A \boxtimes W) \cong (A \otimes_{B^{\mathrm{e}}} A) \boxtimes W \to A \boxtimes W$, for any $B$-bimodule $W$. They are subject to the associativity and unit conditions.

As claimed in Exercise 1.7, the natural transformations $\eta$ and $\mu$ are uniquely determined by their components

$$\eta_{|B^{\mathrm{e}}} : B^{\mathrm{e}} \to A \boxtimes {_{|}B^{\mathrm{e}}} \cong A \qquad \text{and} \qquad \mu_{|B^{\mathrm{e}}} : A \otimes_{B^{\mathrm{e}}} A \cong (A \otimes_{B^{\mathrm{e}}} A) \boxtimes {_{|}B^{\mathrm{e}}} \to A \boxtimes {_{|}B^{\mathrm{e}}} \cong A$$

which are $B^{\mathrm{e}}$-bimodule maps. The associativity and unit conditions on the natural transformations $\eta$ and $\mu$ translate to the commutative diagrams

$$\begin{tikzcd} A \otimes_{B^{\mathrm{e}}} A \otimes_{B^{\mathrm{e}}} A \arrow[r, "\mu_{|B^{\mathrm{e}}} \otimes_{B^{\mathrm{e}}} 1"] \arrow[d, "1 \otimes_{B^{\mathrm{e}}} \mu_{|B^{\mathrm{e}}}"'] & A \otimes_{B^{\mathrm{e}}} A \arrow[d, "\mu_{|B^{\mathrm{e}}}"] \\ A \otimes_{B^{\mathrm{e}}} A \arrow[r, "\mu_{|B^{\mathrm{e}}}"'] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\eta_{|B^{\mathrm{e}}} \otimes_{B^{\mathrm{e}}} 1"] \arrow[d, "1 \otimes_{B^{\mathrm{e}}} \eta_{|B^{\mathrm{e}}}"'] \arrow[dr, equal] & A \otimes_{B^{\mathrm{e}}} A \arrow[d, "\mu_{|B^{\mathrm{e}}}"] \\ A \otimes_{B^{\mathrm{e}}} A \arrow[r, "\mu_{|B^{\mathrm{e}}}"'] & A \end{tikzcd}$$

Given these maps $\eta_{|B^e}$ and $\mu_{|B^e}$, we define a multiplication on $A$ as the composite of the projection $A \otimes A \to A \otimes_{B^e} A$ with $\mu_{|B^e} : A \otimes_{B^e} A \to A$. By the associativity of $\mu_{|B^e}$ it is associative and possesses a unit given by the image of the unit of $B^e$ under $\eta_{|B^e}$. For this algebra structure $\eta_{|B^e} : B^e \to A$ is an algebra homomorphism which induces the given $B^e$-actions on $A$.

Conversely, if the data in part (ii) are given, then the associativity of the algebra $A$ implies that the multiplication of $A$ factorizes through $A \otimes_{B^e} A$ via some $B^e$-bilinear associative multiplication $A \otimes_{B^e} A \to A$. The given algebra homomorphism $B^e \to A$ will be its unit by the assumption that it induces the $B^e$-actions on $A$.

This clearly gives a bijection between the data in parts (i) and (ii).

Concerning the final claim, an Eilenberg–Moore algebra of the monad $A \boxtimes - : \mathsf{bim}(B) \to \mathsf{bim}(B)$ in (i) is a $B$-bimodule $W$ together with an associative and unital action $w : A \boxtimes W \to W$. Then $W$ is an $A$-module via the action provided by the composite of the canonical epimorphism $A \otimes W \to A \boxtimes W$ and $w : A \boxtimes W \to W$.

Conversely, if $V$ is a module over the algebra $A$ then the algebra homomorphism $B^e \to A$ induces a $B^e$-action on $V$ which may be seen as a $B$-bimodule structure, see Paragraph 3.4. Then the given action $A \otimes V \to V$ clearly factorizes through the epimorphism $A \otimes V \to A \boxtimes V$ via the desired action $A \boxtimes V \to V$.

Together with the identity map on the morphisms, these constructions yield the stated mutually inverse functors. □

Recall from part (3) of Examples 2.2 that the category $\mathsf{bim}(B)$ of bimodules over any algebra $B$ is monoidal. Next we wonder about the possible opmonoidal structures on the functor $C \boxtimes - : \mathsf{bim}(B) \to \mathsf{bim}(B)$ induced by a $B^e$-bimodule $C$; see Paragraph 3.4.

**Definition 3.6.** [36] For an arbitrary algebra $B$, a *$B|B$-coring* consists of

- a $B^e$-bimodule $C$
- for the $B$-bimodule $_|C$ of (3.3), a $B$-bimodule map $\Delta : {}_|C \to {}_|C \otimes_B {}_|C$, $c \mapsto c_1 \otimes_B c_2$ (where implicit summation is understood)
- a $B$-bimodule map $\varepsilon : {}_|C \to B$

such that the following conditions hold.

(a) $\Delta$ is a coassociative comultiplication; that is, for any $c \in C$, $c_{11} \otimes_B c_{12} \otimes_B c_2 = c_1 \otimes_B c_{21} \otimes_B c_{22}$.
(b) $\varepsilon$ is the counit of $\Delta$; that is, for any $c \in C$, $(\varepsilon(c_1) \otimes 1) \cdot c_2 = c = (1 \otimes \varepsilon(c_2)) \cdot c_1$.
(c) $\Delta$ respects the further $B$-actions as well; in the sense that for any $c \in C$ and $b \otimes b' \in B \otimes B^{\mathsf{op}}$, $\Delta(c \cdot (b \otimes b')) = c_1 \cdot (b \otimes 1) \otimes c_2 \cdot (1 \otimes b')$.
(d) The image of $\Delta$ is central in a suitable $B$-bimodule; concretely, for any $c \in C$ and $b \in B$, $c_1 \cdot (1 \otimes b) \otimes_B c_2 = c_1 \otimes_B c_2 \cdot (b \otimes 1)$.
(e) The counit $\varepsilon$ satisfies $\varepsilon(c \cdot (b \otimes 1)) = \varepsilon(c \cdot (1 \otimes b))$ for all $c \in C$ and $b \in B$.

A *morphism of $B|B$-corings* is a $B^e$-bimodule map $f : C \to C'$ such that $\varepsilon'(f(c)) = \varepsilon(c)$ and $f(c)_1 \otimes_B f(c)_2 = f(c_1) \otimes_B f(c_2)$ for all $c \in C$.

**Proposition 3.7.** *(1) For any algebra $B$ and any $B^e$-bimodule $C$, there is a bijective correspondence between the following structures.*

(i) *opmonoidal structures on the functor $C \boxtimes - : \mathsf{bim}(B) \to \mathsf{bim}(B)$ (see Paragraph 3.4)*
(ii) *$B|B$-coring structures on the $B^e$-bimodule $C$*

*(2) For two $B|B$-corings $C$ and $C'$ and a $B^{\mathrm{e}}$-bimodule map $f : C \to C'$, the following are equivalent.*

(i) *For all $B$-bimodules $V$, the maps $f \boxtimes 1 : C \boxtimes V \to C' \boxtimes V$ are the components of an opmonoidal natural transformation $C \boxtimes - \to C' \boxtimes -$ between the opmonoidal functors in part (1).*

(ii) *$f$ is a homomorphism of $B|B$-corings.*

*Proof.* (1) First let a $B|B$-coring structure as in (ii) be given. The nullary part $t^0$ of the desired opmonoidal structure in (i) has the domain

$$\begin{aligned} C \boxtimes B &\cong C \otimes B / \{c \cdot (b \otimes b') \otimes p - c \otimes bpb' | c \in C,\ p \in B,\ b \otimes b' \in B^{\mathrm{e}}\} \\ &\cong C / \{c \cdot (b \otimes 1) - c \cdot (1 \otimes b) | c \in C,\ b \in B\}. \end{aligned}$$

By property (e), the counit factorizes through this quotient of $C$ via some $B$-bimodule map $t^0 : C \boxtimes B \to B$.

The binary part $t^2$ should have components of the form

$$t^2_{V,W} : C \boxtimes (V \otimes_B W) \to (C \boxtimes V) \otimes_B (C \boxtimes W)$$

for any $B$-bimodules $V$ and $W$. We claim that it is meaningful to put

$$t^2_{V,W}(c \boxtimes (v \otimes_B w)) = (c_1 \boxtimes v) \otimes_B (c_2 \boxtimes w); \tag{3.4}$$

all the needed balancing conditions hold. To this end consider the (obviously well-defined) map

$$C \otimes C \otimes V \otimes W \to (C \boxtimes V) \otimes_B (C \boxtimes W), \qquad c \otimes c' \otimes v \otimes w \mapsto (c \boxtimes v) \otimes_B (c' \boxtimes w). \tag{3.5}$$

For any $c, c' \in C$, $v \in V$, $w \in W$ and $b \in B$ it satisfies

$$\begin{aligned} ((1 \otimes b) \cdot c \boxtimes v) \otimes_B (c' \boxtimes w) &= (c \boxtimes v) \cdot b \otimes_B (c' \boxtimes w) \\ &= (c \boxtimes v) \otimes_B b \cdot (c' \boxtimes w) = (c \boxtimes v) \otimes_B ((b \otimes 1) \cdot c' \boxtimes w). \end{aligned}$$

In the first and the last equalities we used the forms of the $B$-actions on $C \boxtimes V$ and $C \boxtimes W$, respectively, and the second equality holds by the definition of the $B$-module tensor product. This proves that (3.5) factorizes through the map

$$(|C \otimes_B |C) \otimes V \otimes W \to (C \boxtimes V) \otimes_B (C \boxtimes W), \qquad (c \otimes_B c') \otimes v \otimes w \mapsto (c \boxtimes v) \otimes_B (c' \boxtimes w).$$

Pre-composing it with $\Delta \otimes 1 \otimes 1$, we get the map

$$C \otimes V \otimes W \to (C \boxtimes V) \otimes_B (C \boxtimes W), \qquad c \otimes v \otimes w \mapsto (c_1 \boxtimes v) \otimes_B (c_2 \boxtimes w). \tag{3.6}$$

It satisfies further equalities for any $c \in C$, $v \in V$, $w \in W$ and $b, b' \in B$. First,

$$\begin{aligned} ((c \cdot (b \otimes b'))_1 \boxtimes v) \otimes_B ((c \cdot (b \otimes b'))_2 \boxtimes w) &= (c_1 \cdot (b \otimes 1) \boxtimes v) \otimes_B (c_2 \cdot (1 \otimes b') \boxtimes w) \\ &= (c_1 \boxtimes b \cdot v) \otimes_B (c_2 \boxtimes w \cdot b'). \end{aligned}$$

The first equality follows by axiom (c) in Definition 3.6 and the second one holds by the definition of the module tensor product $\boxtimes$. Furthermore,

$$\begin{aligned} (c_1 \boxtimes v \cdot b) \otimes_B (c_2 \boxtimes w) &= (c_1 \cdot (1 \otimes b) \boxtimes v) \otimes_B (c_2 \boxtimes w) \\ &= (c_1 \boxtimes v) \otimes_B (c_2 \cdot (b \otimes 1) \boxtimes w) = (c_1 \boxtimes v) \otimes_B (c_2 \boxtimes b \cdot w). \end{aligned}$$

The first and the last equalities hold by the definition of the module tensor product $\boxtimes$ and the second one holds by axiom (d) in Definition 3.6. The last two computations prove that

(3.6) factorizes through the well-defined map of (3.4). The map of (3.4) is a $B$-bimodule map since $\Delta$ is so:

$$\begin{aligned}(((b\otimes b')\cdot c)_1 \boxtimes v)\otimes_B (((b\otimes b')\cdot c)_2 \boxtimes w) &= ((b\otimes 1)\cdot c_1 \boxtimes v)\otimes_B ((1\otimes b')\cdot c_2 \boxtimes w)\\ &= b\cdot (c_1\boxtimes v)\otimes_B (c_2\boxtimes w)\cdot b'.\end{aligned}$$

The diagrams of Definition 2.5 commute by axioms (a) and (b) in Definition 3.6.

Conversely, let us be given an opmonoidal structure $(t^0, t^2)$ as in part (i). The to-be-counit $\varepsilon$ is defined as the composite of the $B$-bimodule epimorphism

$$|C \to C/\{c\cdot(b\otimes 1) - c\cdot(1\otimes b)\} \cong C\boxtimes B$$

with $t^0 : C\boxtimes B \to B$. By construction it is a $B$-bimodule map satisfying axiom (e) in Definition 3.6.

The to-be-comultiplication $\Delta$ is defined as the composite of the $B$-bimodule map

$$|C \to C\boxtimes (|B^{\mathrm{e}} \otimes_B |B^{\mathrm{e}}), \qquad\qquad c\mapsto c\boxtimes ((1\otimes 1)\otimes_B (1\otimes 1)) \tag{3.7}$$

with $t^2_{|B^{\mathrm{e}},|B^{\mathrm{e}}} : C\boxtimes (|B^{\mathrm{e}}\otimes_B |B^{\mathrm{e}}) \to (C\boxtimes |B^{\mathrm{e}})\otimes_B (C\boxtimes |B^{\mathrm{e}})$ and with the $B$-module tensor product of the $B$-bimodule isomorphisms $C\boxtimes |B^{\mathrm{e}} \cong |C$. Then it is a $B$-bimodule map by construction. Denote the image of any element $c$ of $C$ under this map $\Delta$ by $c_1\otimes_B c_2$ (where implicit summation is understood).

We turn to checking the validity of the axioms of Definition 3.6. Consider the $B$-bimodule map

$$-\cdot v\cdot - : |B^{\mathrm{e}} \to V, \qquad b\otimes b' \mapsto b\cdot v\cdot b' \tag{3.8}$$

induced by any element $v$ of an arbitrary $B$-bimodule $V$. By the naturality of $t^2$ the diagram

$$\begin{array}{ccc} C\boxtimes (|B^{\mathrm{e}}\otimes_B |B^{\mathrm{e}}) & \xrightarrow{t^2_{|B^{\mathrm{e}},|B^{\mathrm{e}}}} & (C\boxtimes |B^{\mathrm{e}})\otimes_B (C\boxtimes |B^{\mathrm{e}}) \\ {\scriptstyle 1\boxtimes(-\cdot v\cdot - \otimes_B - \cdot w\cdot -)}\downarrow & & \downarrow{\scriptstyle (1\boxtimes - \cdot v\cdot -)\otimes_B (1\boxtimes -\cdot w\cdot -)} \\ C\boxtimes (V\otimes_B W) & \xrightarrow[t^2_{V,W}]{} & (C\boxtimes V)\otimes_B (C\boxtimes W) \end{array}$$

commutes. Its left-bottom path takes $c\boxtimes((1\otimes 1)\otimes_B (1\otimes 1))$ (for any $c\in C$) to $t^2_{V,W}(c\boxtimes(v\otimes_B w))$; while the top-right path takes it to $(c_1\boxtimes v)\otimes_B (c_2\boxtimes w)$. This proves

$$t^2_{V,W}(c\boxtimes (v\otimes_B w)) = (c_1\boxtimes v)\otimes_B (c_2\boxtimes w) \tag{3.9}$$

for any elements $c$ of $C$, $v$ of an arbitrary $B$-bimodule $V$ and $w$ of an arbitrary $B$-bimodule $W$.

With (3.9) at hand, axioms (a) and (b) of Definition 3.6 hold by the commutativity of the diagrams of Definition 2.5; taking each of the occurring objects equal to $|B^{\mathrm{e}}$.

The final two axioms (c) and (d) follow by the naturality of $t^2$ as we claim next. Consider the $B$-bimodule maps of (3.8) induced by the particular elements $b\otimes 1$ and $1\otimes b$ of the $B$-bimodule $|B^{\mathrm{e}}$, for an arbitrary fixed element $b$ of $B$. By the naturality of $t^2$ the following diagram commutes.

$$\begin{tikzcd}
C\boxtimes(|B^{\mathsf{e}}\otimes_B |B^{\mathsf{e}}) \arrow[r, "t^2_{|B^{\mathsf{e}},|B^{\mathsf{e}}}"] \arrow[d, "1\boxtimes(-\cdot(b\otimes 1)\cdot-\otimes_B-\cdot(1\otimes b')\cdot-)"'] & (C\boxtimes|B^{\mathsf{e}})\otimes_B(C\boxtimes|B^{\mathsf{e}}) \arrow[r, "\cong"] \arrow[d, "(1\boxtimes-\cdot(b\otimes 1)\cdot-)\otimes_B(1\boxtimes-\cdot(1\otimes b')\cdot-)"'] & C\otimes_B C \arrow[d, "-\cdot(b\otimes 1)\otimes_B-\cdot(1\otimes b')"'] \\
C\boxtimes(|B^{\mathsf{e}}\otimes_B |B^{\mathsf{e}}) \arrow[r, "t^2_{|B^{\mathsf{e}},|B^{\mathsf{e}}}"'] & (C\boxtimes|B^{\mathsf{e}})\otimes_B(C\boxtimes|B^{\mathsf{e}}) \arrow[r, "\cong"'] & C\otimes_B C
\end{tikzcd}$$

For any $c\in C$, the value of the left vertical on $c\boxtimes((1\otimes 1)\otimes_B(1\otimes 1))$ is

$$c\boxtimes((b\otimes 1)\otimes_B(1\otimes b'))=c\cdot(b\otimes b')\boxtimes((1\otimes 1)\otimes_B(1\otimes 1)).$$

Hence the left-bottom path takes it to $\Delta(c\cdot(b\otimes b'))$; while the top-right path takes it to $c_1\cdot(b\otimes 1)\otimes c_2\cdot(1\otimes b')$. This proves axiom (c). Similarly, the naturality of $t^2$ implies commutativity of the following diagrams as well.

$$\begin{tikzcd}
C\boxtimes(|B^{\mathsf{e}}\otimes_B |B^{\mathsf{e}}) \arrow[r, "t^2_{|B^{\mathsf{e}},|B^{\mathsf{e}}}"] \arrow[d, "1\boxtimes(-\cdot(1\otimes b)\cdot-\otimes_B 1)"'] & (C\boxtimes|B^{\mathsf{e}})\otimes_B(C\boxtimes|B^{\mathsf{e}}) \arrow[r, "\cong"] \arrow[d, "(1\boxtimes-\cdot(1\otimes b)\cdot-)\otimes_B 1"] & C\otimes_B C \arrow[d, "-\cdot(1\otimes b)\otimes_B 1"] \\
C\boxtimes(|B^{\mathsf{e}}\otimes_B |B^{\mathsf{e}}) \arrow[r, "t^2_{|B^{\mathsf{e}},|B^{\mathsf{e}}}"'] & (C\boxtimes|B^{\mathsf{e}})\otimes_B(C\boxtimes|B^{\mathsf{e}}) \arrow[r, "\cong"'] & C\otimes_B C
\end{tikzcd}$$

$$\begin{tikzcd}
C\boxtimes(|B^{\mathsf{e}}\otimes_B |B^{\mathsf{e}}) \arrow[r, "t^2_{|B^{\mathsf{e}},|B^{\mathsf{e}}}"] \arrow[d, "1\boxtimes(1\otimes_B-\cdot(b\otimes 1)\cdot-)"'] & (C\boxtimes|B^{\mathsf{e}})\otimes_B(C\boxtimes|B^{\mathsf{e}}) \arrow[r, "\cong"] \arrow[d, "1\otimes_B(1\boxtimes-\cdot(b\otimes 1)\cdot-)"] & C\otimes_B C \arrow[d, "1\otimes_B-\cdot(b\otimes 1)"] \\
C\boxtimes(|B^{\mathsf{e}}\otimes_B |B^{\mathsf{e}}) \arrow[r, "t^2_{|B^{\mathsf{e}},|B^{\mathsf{e}}}"'] & (C\boxtimes|B^{\mathsf{e}})\otimes_B(C\boxtimes|B^{\mathsf{e}}) \arrow[r, "\cong"'] & C\otimes_B C
\end{tikzcd}$$

Their left verticals take $c\boxtimes((1\otimes 1)\otimes_B(1\otimes 1))$ (for any $c\in C$) to the equal elements

$$c\boxtimes((1\otimes b)\otimes_B(1\otimes 1))=c\boxtimes((1\otimes 1)\otimes_B(b\otimes 1)),$$

respectively, hence their left-bottom paths are equal on that element. Then so must be the top-right paths proving axiom (d).

It remains to see that the above constructions are mutual inverses. Starting with a $B|B$-coring in part (ii) and iterating these constructions we evidently re-obtain the original $B|B$-coring. Starting with an opmonoidal structure $(t^0,t^2)$ in part (i) and iterating the constructions in the opposite order, we obtain an opmonoidal functor whose nullary part is clearly the same map $t^0$. The binary parts agree by (3.9).

(2) Opmonoidality of the natural transformation with components $f\boxtimes 1: C\boxtimes V\to C'\boxtimes V$ (for any $B$-bimodule $V$) translates to the commutativity of the diagrams

$$\begin{tikzcd}
C\boxtimes B \arrow[r, "t^0"] \arrow[d, "f\boxtimes 1"'] & B \arrow[d, equal] \\
C'\boxtimes B \arrow[r, "t'^0"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
C\boxtimes(V\otimes_B W) \arrow[r, "t^2_{V,W}"] \arrow[d, "f\boxtimes 1"'] & (C\boxtimes V)\otimes_B(C\boxtimes W) \arrow[d, "(f\boxtimes 1)\otimes_B(f\boxtimes 1)"] \\
C'\boxtimes(V\otimes_B W) \arrow[r, "t'^2_{V,W}"'] & (C'\boxtimes V)\otimes_B(C'\boxtimes W)
\end{tikzcd}$$

for any $B$-bimodules $V$ and $W$.

Pre-composing the equal paths around the first one with the canonical epimorphism $C \to C \boxtimes B$ it gives the equivalent condition $\varepsilon' \circ f = \varepsilon$.

The top-right path of the second diagram sends an element $c \boxtimes (v \otimes_B w)$ of the domain to $(f(c_1) \boxtimes v) \otimes_B (f(c_2) \boxtimes w)$ while left bottom path sends it to $(f(c)_1 \boxtimes v) \otimes_B (f(c)_2 \boxtimes w)$. Hence if $f$ satisfies the comultiplicativity condition $f(c)_1 \otimes_B f(c)_2 = f(c_1) \otimes_B f(c_2)$ then the second diagram commutes. The opposite implication follows by evaluating the diagram at $V = W = {}_{|}B^{\mathsf{e}}$ and pre-composing its equal paths with the map of (3.7) and post-composing them in both factors with the isomorphism $C \boxtimes {}_{|}B^{\mathsf{e}} \cong C$. $\square$

**Corollary 3.8.** *(1) For any algebra $B$, the $B^{\mathsf{e}}$-bimodule $B^{\mathsf{e}}$ (with actions provided by the multiplication) carries a $B|B$-coring structure with comultiplication*

$$B^{\mathsf{e}} \to B^{\mathsf{e}} \otimes_B B^{\mathsf{e}}, \qquad b \otimes b' \mapsto (b \otimes 1) \otimes_B (1 \otimes b')$$

*and counit*

$$B^{\mathsf{e}} \to B, \qquad b \otimes b' \mapsto bb'.$$

*(2) For any $B|B$-corings $C$ and $C'$, the $B^{\mathsf{e}}$-module tensor product $C \otimes_{B^{\mathsf{e}}} C'$ is a $B|B$-coring with comultiplication*

$$c \otimes_{B^{\mathsf{e}}} c' \mapsto (c_1 \otimes_{B^{\mathsf{e}}} c'_1) \otimes_B (c_2 \otimes_{B^{\mathsf{e}}} c'_2)$$

*and counit*

$$c \otimes_{B^{\mathsf{e}}} c' \mapsto \varepsilon'(c' \cdot (\varepsilon(c) \otimes 1)) \equiv \varepsilon'(c' \cdot (1 \otimes \varepsilon(c))).$$

*Proof.* (1) The functor $B^{\mathsf{e}} \boxtimes -$ is naturally isomorphic to the identity functor $\mathsf{bim}(B) \to \mathsf{bim}(B)$; which is opmonoidal by part (1) of Examples 2.6. An easy computation shows that via the correspondence in part (1) of Proposition 3.7, the opmonoidal functor $B^{\mathsf{e}} \boxtimes - : \mathsf{bim}(B) \to \mathsf{bim}(B)$ corresponds to the $B|B$-coring in the claim.

(2) Both $B|B$-corings in the claim determine opmonoidal functors $C \boxtimes -$ and $C' \boxtimes - : \mathsf{bim}(B) \to \mathsf{bim}(B)$ as in part (1) of Proposition 3.7. Their composite $C \boxtimes (C' \boxtimes -) \cong (C \otimes_{B^{\mathsf{e}}} C') \boxtimes -$ is opmonoidal by Exercise 2.7. An easy computation shows again that via the correspondence in part (1) of Proposition 3.7, it corresponds to the $B|B$-coring in the claim. $\square$

**Theorem & Definition 3.9.** [32] *For any algebra $B$ and any $B^{\mathsf{e}}$-bimodule $T$, there is a bijective correspondence between the following structures.*

(i) *bimonad structures on the functor $T \boxtimes - : \mathsf{bim}(B) \to \mathsf{bim}(B)$ (see Paragraph 3.4)*

(ii) *a $B^{\mathsf{e}}$-ring structure $\eta$, and a $B|B$-coring structure $(\Delta, \varepsilon)$ on the $B^{\mathsf{e}}$-bimodule $T$ such that $\eta$ and the projected multiplication map $T \otimes_{B^{\mathsf{e}}} T \to T$ are morphisms of $B|B$-corings. That is, the following identities hold.*

  (a) *for any $c, c' \in T$, $\Delta(cc') = c_1 c'_1 \otimes_B c_2 c'_2$ (note that the right hand side is meaningful by axiom (d) of Definition 3.6)*

  (b) *for the unit $1_T$ of the algebra $T$, $\Delta(1_T) = 1_T \otimes_B 1_T$*

  (c) *for any $c, c' \in T$, $\varepsilon(cc') = \varepsilon(c\eta(1_B \otimes \varepsilon(c'))) \equiv \varepsilon(c\eta(\varepsilon(c') \otimes 1_B))$ (the equality of the last two expressions follows by axiom (e) of Definition 3.6)*

  (d) *for the unit $1_T$ of the algebra $T$ and the unit $1_B$ of the algebra $B$, $\varepsilon(1_T) = 1_B$.*

*The structure in part (ii) is called a $B$-*bialgebroid *structure on $T$ (see* [20]*). A slightly different, but equivalent formulation in* [35] *was also called a $\times_B$-*bialgebra*.*

*Furthermore, the bimonad $T \boxtimes - : \mathsf{bim}(B) \to \mathsf{bim}(B)$ in part (i) is a Hopf monad if and only if the map from the $B^{\mathsf{op}}$-module tensor product*

$$T \otimes_{B^{\mathsf{op}}} T := T \otimes T / \{c\eta(1_B \otimes b) \otimes c' - c \otimes \eta(1_B \otimes b)c' | c, c' \in T, \ b \in B\}$$

*to the B-module tensor product*

$$T\otimes_B T := T\otimes T/\{\eta(1_B\otimes b)c\otimes c' - c\otimes\eta(b\otimes 1_B)c' | c,c'\in C,\ b\in B\}$$

*sending $c\otimes_{B^{\mathrm{op}}} c'$ to $c_1\otimes_B c_2c'$ is invertible (note that it is well-defined by axiom (c) in Definition 3.6). In this case $T$ is called a* (Schauenburg) Hopf algebroid *over $B$ (or $\times_B$-*Hopf algebra*, by other authors).*

*Proof.* The bijective correspondence between the data in parts (i) and (ii) follows immediately from Proposition 3.5 and Proposition 3.7.

For the bimonad $T\boxtimes - : \mathsf{bim}(B)\to\mathsf{bim}(B)$ in part (i), the components of the natural transformation in part (iii) of Theorem & Definition 2.18 take the following form, for any $B$-bimodules $V$ and $W$, $v\in V$, $w\in W$ and $c,c'\in T$.

$$T\boxtimes(V\otimes_B(T\boxtimes W))\to(T\boxtimes V)\otimes_B(T\boxtimes W),\quad c\boxtimes(v\otimes_B(c'\boxtimes w))\mapsto(c_1\boxtimes v)\otimes_B(c_2c'\boxtimes w) \tag{3.10}$$

This is clearly invertible if the stated map

$$T\otimes_{B^{\mathrm{op}}} T\to T\otimes_B T,\qquad c\otimes_{B^{\mathrm{op}}} c'\mapsto c_1\otimes_B c_2c' \tag{3.11}$$

is so. Conversely, if (3.10) is invertible for any $B$-bimodules $V$ and $W$ then it is invertible in particular for $V=W={}_{|}B^{\mathrm{e}}$. Composing this isomorphism with the isomorphisms

$$\begin{aligned}&T\boxtimes({}_{|}B^{\mathrm{e}}\otimes_B(T\boxtimes{}_{|}B^{\mathrm{e}}))\to T\otimes_{B^{\mathrm{op}}} T,\\ &c\boxtimes((p\otimes p')\otimes_B(c'\boxtimes(b\otimes b')))\mapsto c\eta(p\otimes 1_B)\otimes_{B^{\mathrm{op}}}\eta(p'\otimes 1_B)c'\eta(b\otimes b')\end{aligned}$$

and

$$\begin{aligned}&(T\boxtimes{}_{|}B^{\mathrm{e}})\otimes_B(T\boxtimes{}_{|}B^{\mathrm{e}})\to T\otimes_B T,\\ &(c\boxtimes(p\otimes p'))\otimes_B(c'\boxtimes(b\otimes b'))\mapsto c\eta(p\otimes p')\otimes_B c\eta(b\otimes b')\end{aligned}$$

we obtain the isomorphism (3.11). □

Let us emphasize that for general Hopf algebroids $T$ there is no antipode of the type $T\to T$.

The notion of Hopf algebroid in Theorem & Definition 3.9 should not be mixed up with an inequivalent definition in [20] under the same name.

**Example 3.10.** For any algebra $B$, the enveloping algebra $B^{\mathrm{e}}$ is a $B^{\mathrm{e}}$-bimodule via left and right multiplication. The induced functor $B^{\mathrm{e}}\boxtimes - : \mathsf{bim}(B)\to\mathsf{bim}(B)$ is naturally isomorphic to the identity functor which carries a trivial structure of opmonoidal monad. Furthermore, for the identity functor $\mathsf{bim}(B)\to\mathsf{bim}(B)$ the components of the natural transformation in part (iii) of Theorem & Definition 2.18 are identity morphisms thus they are invertible. This says that the identity functor $\mathsf{bim}(B)\to\mathsf{bim}(B)$ — and hence $B^{\mathrm{e}}\boxtimes - : \mathsf{bim}(B)\to\mathsf{bim}(B)$ which is naturally isomorphic to it — can be regarded as a Hopf monad. Then by the application of Theorem & Definition 3.9 we infer a Hopf algebroid structure of $B^{\mathrm{e}}$. Explicitly, the $B^{\mathrm{e}}$-ring structure is given by the identity map $\eta : B^{\mathrm{e}}\to B^{\mathrm{e}}$ while the comultiplication $\Delta$ and the counit $\varepsilon$ take the respecive forms

$$\Delta(b\otimes b')=(b\otimes 1)\otimes_B(1\otimes b')\qquad \varepsilon(b\otimes b')=bb'\qquad \text{for all } b\otimes b'\in B^{\mathrm{e}}.$$

There is an extended literature on bialgebroids and Hopf algebroids, see e.g. [32] [33] [26] [27] [28] [17] [16] [19] [18] [3] [13] [14] [4] [2] [7] [5] [11] [6]. Many results on classical bialgebras and Hopf algebras have been extended to them. But there is more than that: they allowed for answers to questions that could not be settled within the classical theory.

As an illustration, or rather advertisement, let us mention — without entering any details — the following highlight result due to Szlachányi and his collaborators. As it is well known, a Galois field extension with some group has an inherent characterization without explicit mention of the Galois group: it is a normal and separable field extension. Galois extensions of algebras by Hopf algebras widely generalize Galois field extensions. They received great attention, among other reasons by their application in non-commutative differential geometry, in the description of non-commutative fibered bundles. However, no inherent characterization of Galois extensions of algebras by Hopf algebras is known, without explicit reference to the Hopf algebra in question. As a remarkable achievement, such a description is available for Galois extensions of algebras by certain, finitary (Hopf) bialgebroids:

**Theorem 3.11.** [19][3] *For an algebra extension $N \subseteq M$, the following assertions are equivalent.*

(i) *$N \subseteq M$ is a Galois extension by some finitely generated and projective bialgebroid.*
(ii) *The algebra extension $N \subseteq M$ satisfies the so-called balancing and depth 2 conditions, see* [19].

*The following assertions are equivalent to each other too.*

(i') *$N \subseteq M$ is a Galois extension by some Frobenius Hopf algebroid, see* [3].
(ii') *In addition to the balancing and depth 2 conditions in part (ii), $N \subseteq M$ is a Frobenius extension.*

## 4. Lecture: Weak (Hopf) bialgebras

The fourth lecture is devoted to the particular (Hopf) bialgebroids whose base algebra possesses a separable Frobenius structure; known as *weak (Hopf) bialgebras*. The basic references are [10], [34] and [29].

**Definition 4.1.** A *separable Frobenius structure* on a functor $f : \mathsf{A}' \to \mathsf{A}$ between monoidal categories consists of

- a monoidal structure $(p^0 : I \to fI', p^2 : f- \otimes f- \to f(- \otimes' -))$
- an opmonoidal structure $(i^0 : fI' \to I, i^2 : f(- \otimes' -) \to f- \otimes f-)$

such that for any objects $X, Y$ and $Z$ of the category $\mathsf{A}'$, $p^2_{X,Y} \circ i^2_{X,Y} = 1$ holds and the following diagrams commute.

$$
\begin{tikzcd}
f(X\otimes' Y)\otimes fZ \arrow[r, "p^2_{X\otimes' Y,Z}"] \arrow[d, "i^2_{X,Y}\otimes 1"'] & f(X\otimes' Y\otimes' Z) \arrow[d, "i^2_{X,Y\otimes' Z}"] \\
fX\otimes fY\otimes fZ \arrow[r, "1\otimes p^2_{Y,Z}"'] & fX\otimes f(Y\otimes' Z)
\end{tikzcd}
\qquad
\begin{tikzcd}
fX\otimes f(Y\otimes' Z) \arrow[r, "p^2_{X,Y\otimes' Z}"] \arrow[d, "1\otimes i^2_{Y,Z}"'] & f(X\otimes' Y\otimes' Z) \arrow[d, "i^2_{X\otimes' Y,Z}"] \\
fX\otimes fY\otimes fZ \arrow[r, "p^2_{X,Y}\otimes 1"'] & f(X\otimes' Y)\otimes fZ
\end{tikzcd}
$$

For a separable Frobenius functor $f$ from an arbitrary category $\mathsf{A}'$ to vec, and any objects $X$ and $Y$ of $\mathsf{A}'$, the vector space $f(X \otimes' Y)$ can be identified with the range of the idempotent linear map $i^2_{X,Y} \circ p^2_{X,Y} : fX \otimes fY \to fX \otimes fY$.

**Example 4.2.** A separable Frobenius structure on any strict monoidal functor is provided by the identity natural transformations.

**Exercise 4.3.** Show that the composite of separable Frobenius functors is separable Frobenius too, via the monoidal and opmonoidal structures as in Exercise 2.7.

**Proposition 4.4.** *For any algebra $B$ over a field $k$, there is a bijective correspondence between the following structures.*

(i) *separable Frobenius structures on the monoidal forgetful functor $u : \mathsf{bim}(B) \to \mathsf{vec}$ of part (3) of Examples 2.6*

(ii) *coalgebra structures $(i^0 : B \to k, \delta : B \to B \otimes B)$ on $B$ whose comultiplication is a $B$-bimodule section of the multiplication; that is — denoting by $m : B \otimes B \to B$ the multiplication — the following diagrams commute.*

$$\begin{tikzcd} B\otimes B \arrow[rr, "1\otimes\delta"] \arrow[dr, "m"] \arrow[d, "\delta\otimes 1"'] & & B\otimes B\otimes B \arrow[d, "m\otimes 1"] \\ B\otimes B\otimes B \arrow[rr, "1\otimes m"'] & B \arrow[r, "\delta"] & B\otimes B \end{tikzcd} \qquad \begin{tikzcd} B \arrow[rr, equal] \arrow[dr, "\delta"'] & & B \\ & B\otimes B \arrow[ur, "m"'] & \end{tikzcd}$$

*The structure in part (ii) is called a* separable Frobenius algebra structure *on $B$.*

*Proof.* Assume first that an opmonoidal structure $(i^0, i^2)$ as in part (i) is given. We define the comultiplication $\delta$ as the composite

$$uB \xrightarrow{\cong} u(B \otimes_B B) \xrightarrow{i^2_{B,B}} uB \otimes uB.$$

By the commutativity of the diagrams of Definition 2.5, it is coassociative and possesses the counit $i^0$. By construction it is a section of the multiplication

$$uB \otimes uB \xrightarrow{p^2_{B,B}} u(B \otimes_B B) \xrightarrow{\cong} uB.$$

Evaluating the diagrams of Definition 4.1 at $X = Y = Z = B$ we infer the $B$-bilinearity of $\delta$.

Conversely, let us be given a separable Frobenius algebra structure on $B$ as in (ii). The unit element 1 of $B$ is central in the $B$-bimodule $B$ whose actions are given by multiplication. Since $\delta$ is a $B$-bimodule map, from this $B$-bimodule $B$ to $|B^{\mathrm{e}}$ of (3.2), it takes the central element 1 to a central element $1_{\langle 1\rangle} \otimes 1_{\langle 2\rangle} := \delta(1)$ in $|B^{\mathrm{e}}$ (where implicit summation is understood). Hence for any $B$-bimodules $V$ and $W$ there is a well-defined $B$-bimodule map

$$i^2_{V,W} : u(V \otimes_B W) \to uV \otimes uW, \qquad v \otimes_B w \mapsto v \cdot 1_{\langle 1\rangle} \otimes 1_{\langle 2\rangle} \cdot w.$$

It is immediate to see that together with $i^0 : uB \to k$ (which is the counit of $\delta$) they provide an opmonoidal structure on $u$; that the diagrams of Definition 4.1 commute; and that since $\delta$ is a section of the multiplication, $p^2 \circ i^2$ is the identity natural transformation. Thus we constructed a separable Frobenius structure on the monoidal functor $u : \mathsf{bim}(B) \to \mathsf{vec}$.

Starting with the structure in part (ii) and iterating the above constructions we clearly re-obtain the original data. Starting with the data $(i^0, i^2)$ in part (i) and iterating the above constructions in the opposite order we obtain an opmonoidal structure evidently with the same nullary part $i^0$. It is more involved to see that we also re-obtain the original binary part $i^2$. Consider the $B$-bimodule $|B^{\mathrm{e}}$ of (3.2) and the $B$-bimodule maps of (3.8). If we denote (understanding implicit summation) by $1_{[1]} \otimes 1_{[2]} \otimes 1_{[3]} \otimes 1_{[4]} \in B \otimes B \otimes B \otimes B$ the value of

$i^2_{|B^e,|B^e}$ on $(1\otimes 1)\otimes_B(1\otimes 1)$, then the naturality of $i^2$, more concretely, commutativity of the diagram

$$\begin{tikzcd}
{}_{|}B^{\mathrm{e}}\otimes_B {}_{|}B^{\mathrm{e}} \arrow[r, "i^2_{|B^e,|B^e}"] \arrow[d, "(-\cdot v\cdot -)\otimes_B(-\cdot w\cdot -)"'] & {}_{|}B^{\mathrm{e}}\otimes {}_{|}B^{\mathrm{e}} \arrow[d, "(-\cdot v\cdot -)\otimes(-\cdot w\cdot -)"] \\
V\otimes_B W \arrow[r, "i^2_{V,W}"'] & V\otimes W
\end{tikzcd}$$

implies that for any $v\otimes_B w\in V\otimes_B W$,

$$i^2_{V,W}(v\otimes_B w)=1_{[1]}\cdot v\cdot 1_{[2]}\otimes 1_{[3]}\cdot w\cdot 1_{[4]}. \tag{4.1}$$

Evaluate now the diagrams of Definition 4.1, for $X=Y=Z={}_{|}B^{\mathrm{e}}$, on the elements $((1\otimes 1)\otimes_B(1\otimes 1))\otimes 1\otimes 1$ and $1\otimes 1\otimes((1\otimes 1)\otimes_B(1\otimes 1))$, respectively. Using (4.1), it yields the respective conditions

$$1_{[1]}\otimes 1_{[2]}\otimes((1_{[3]}\otimes 1)\otimes_B(1\otimes 1_{[4]}))=1_{[1]}\otimes 1_{[2]}\otimes((1_{[3]}\otimes 1_{[4]})\otimes_B(1\otimes 1)) \qquad \text{and}$$
$$((1_{[1]}\otimes 1)\otimes_B(1\otimes 1_{[2]}))\otimes 1_{[3]}\otimes 1_{[4]}=((1\otimes 1)\otimes_B(1_{[1]}\otimes 1_{[2]}))\otimes 1_{[3]}\otimes 1_{[4]}.$$

Multiplying the third, fourth and fifth tensorands of the first equality, and multiplying the second, third and fourth tensorands of the second equality, this implies

$$1\otimes 1_{[1]}1_{[2]}\otimes 1_{[3]}\otimes 1_{[4]}=1_{[1]}\otimes 1_{[2]}\otimes 1_{[3]}\otimes 1_{[4]}=1_{[1]}\otimes 1_{[2]}\otimes 1_{[3]}1_{[4]}\otimes 1$$

and hence

$$1_{[1]}\otimes 1_{[2]}\otimes 1_{[3]}\otimes 1_{[4]}=1\otimes 1_{[1]}1_{[2]}\otimes 1_{[3]}1_{[4]}\otimes 1. \tag{4.2}$$

On the other hand, the multiplication is a $B$-bimodule map $m: {}_{|}B^{\mathrm{e}}\to B$. So by the naturality of $i^2$ also the following diagram commutes

$$\begin{tikzcd}
{}_{|}B^{\mathrm{e}}\otimes_B {}_{|}B^{\mathrm{e}} \arrow[r, "i^2_{|B^e,|B^e}"] \arrow[d, "m\otimes_B m"'] & {}_{|}B^{\mathrm{e}}\otimes {}_{|}B^{\mathrm{e}} \arrow[d, "m\otimes m"] \\
B\otimes_B B \arrow[r, "i^2_{B,B}"'] & B\otimes B
\end{tikzcd}$$

proving $1_{[1]}1_{[2]}\otimes 1_{[3]}1_{[4]}=i^2_{B,B}(1)$ which we denoted earlier by $1_{\langle 1\rangle}\otimes 1_{\langle 2\rangle}$ (understanding implicit summation). Combining this with (4.2) we see that $1_{[1]}\otimes 1_{[2]}\otimes 1_{[3]}\otimes 1_{[4]}=1\otimes 1_{\langle 1\rangle}\otimes 1_{\langle 2\rangle}\otimes 1$ and thus from (4.1), $i^2_{V,W}(v\otimes_B w)=v\cdot 1_{\langle 1\rangle}\otimes 1_{\langle 2\rangle}\cdot w$ as needed. □

We deduce from Proposition 4.4 that for a separable Frobenius algebra $B$, the $B$-module tensor product $V\otimes_B W$ (of any $B$-bimodules $V$ and $W$) can be identified with the image of the idempotent map

$$V\otimes W\to V\otimes W, \qquad v\otimes w\mapsto v\cdot 1_{\langle 1\rangle}\otimes 1_{\langle 2\rangle}\cdot w$$

for the image $1_{\langle 1\rangle}\otimes 1_{\langle 2\rangle}$ of the unit element of $B$ under the comultiplication (using an implicit summation notation).

**Theorem & Definition 4.5.** *For any algebra $A$ over a field $k$, there is a bijective correspondence between the following structures.*

(i)
- *monoidal structures on* $\mathsf{mod}(A)$ *and*
- *separable Frobenius structures on the forgetful functor* $U:\mathsf{mod}(A)\to\mathsf{vec}$

(ii) • *bialgebroid structures on A (over some unspecified base algebra) and*
• *separable Frobenius algebra structures on the base algebra of the bialgebroid*

(iii) *coalgebra structures* $(\widehat{\Delta},\widehat{\varepsilon})$ *on A such that for all* $a,a',a''\in A$ *the identities*
(a) $\widehat{\Delta}(aa')=\widehat{\Delta}(a)\widehat{\Delta}(a')$
(b) $(\widehat{\Delta}(1)\otimes 1)(1\otimes\widehat{\Delta}(1))=1_{\hat 1}\otimes 1_{\hat 2}\otimes 1_{\hat 3}=(1\otimes\widehat{\Delta}(1))(\widehat{\Delta}(1)\otimes 1)$
(c) $\widehat{\varepsilon}(aa'_{\hat 1})\widehat{\varepsilon}(a'_{\hat 2}a'')=\widehat{\varepsilon}(aa'a'')=\widehat{\varepsilon}(aa'_{\hat 2})\widehat{\varepsilon}(a'_{\hat 1}a'')$

*hold, where* $1$ *stands for the unit element of the algebra A and* $a_{\hat 1}\otimes a_{\hat 2}=\widehat{\Delta}(a)$ *for any* $a\in A$ *(where implicit summation is understood).*

*The structure in part (iii) is called a* weak bialgebra *A.*

*Furthermore, the bialgebroid in part (ii) is a Hopf algebroid if and if there is a map* $\sigma:A\to A$ *such that for all* $a\in A$ *the following equalities hold.*

$$a_{\hat 1}\sigma(a_{\hat 2})=\widehat{\varepsilon}(1_{\hat 1}a)1_{\hat 2}\qquad \sigma(a_{\hat 1})a_{\hat 2}=1_{\hat 1}\widehat{\varepsilon}(a1_{\hat 2})\qquad \sigma(a_{\hat 1})a_{\hat 2}\sigma(a_{\hat 3})=\sigma(a)$$

*In this situation the weak bialgebra A is said to be a* weak Hopf algebra *with the* antipode $\sigma$.

Before we start to prove Theorem & Definition 4.5, let us collect some needed technicalities in the form of an exercise.

**Exercise 4.6.** For a weak bialgebra $A$, the map

$$\varepsilon:A\to A,\qquad a\mapsto\widehat{\varepsilon}(1_{\hat 1}a)1_{\hat 2}\tag{4.3}$$

satisfies the following identities for any $a,a'\in A$.

(a) $1_{\hat 1}\otimes\varepsilon(1_{\hat 2})=1_{\hat 1}\otimes 1_{\hat 2}$ so in particular $\varepsilon(1)=1$
(b) $\widehat{\varepsilon}(aa')=\widehat{\varepsilon}(a\varepsilon(a'))$ so in particular $\widehat{\varepsilon}\varepsilon(a)=\widehat{\varepsilon}(a)$
(c) $\varepsilon(aa')=\varepsilon(a\varepsilon(a'))$ so in particular $\varepsilon\varepsilon(a)=\varepsilon(a)$
(d) $\widehat{\Delta}(a\varepsilon(a'))=a_{\hat 1}\varepsilon(a')\otimes a_{\hat 2}$ and
$\widehat{\Delta}(\varepsilon(a')a)=\varepsilon(a')a_{\hat 1}\otimes a_{\hat 2}$ so in particular $1_{\hat 1}\varepsilon(a)\otimes 1_{\hat 2}=\varepsilon(a)1_{\hat 1}\otimes 1_{\hat 2}$
(e) $\varepsilon(\varepsilon(a)a')=\varepsilon(a)\varepsilon(a')$
(f) $\varepsilon(a)=\varepsilon(1_{\hat 1})\widehat{\varepsilon}(1_{\hat 2}a)$
(g) $\varepsilon(a_{\hat 1})\otimes a_{\hat 2}=\varepsilon(1_{\hat 1})\otimes 1_{\hat 2}a$
(h) $\varepsilon(a)\varepsilon(1_{\hat 1})\otimes 1_{\hat 2}=\varepsilon(1_{\hat 1})\otimes 1_{\hat 2}\varepsilon(a)$
(i) $\varepsilon(a_{\hat 1})a_{\hat 2}=a$

*Hint.* (a) $1_{\hat 1}\otimes\varepsilon(1_{\hat 2})=1_{\hat 1}\otimes\widehat{\varepsilon}(1_{\hat 1'}1_{\hat 2})1_{\hat 2'}\overset{\text{Axiom }(b)}{=}1_{\hat 1}\otimes\widehat{\varepsilon}(1_{\hat 2})1_{\hat 3}=1_{\hat 1}\otimes 1_{\hat 2}$ and apply $\widehat{\varepsilon}$ to the first tensor factor.
(b) $\widehat{\varepsilon}(a\varepsilon(a'))=\widehat{\varepsilon}(a1_{\hat 2})\widehat{\varepsilon}(1_{\hat 1}a')\overset{\text{Axiom }(c)}{=}\widehat{\varepsilon}(aa')$ and put $a=1$.
(c) $\varepsilon(aa')=\widehat{\varepsilon}(1_{\hat 1}aa')1_{\hat 2}\overset{(b)}{=}\widehat{\varepsilon}(1_{\hat 1}a\varepsilon(a'))1_{\hat 2}=\varepsilon(a\varepsilon(a'))$ and put $a=1$.

(d) $\widehat{\Delta}(a\varepsilon(a')) = \widehat{\varepsilon}(1_{\hat 1}a')(a1_{\hat 2})_{\hat 1} \otimes (a1_{\hat 2})_{\hat 2} \overset{\text{Axiom }(a)}{=} \widehat{\varepsilon}(1_{\hat 1}a')a_{\hat 1}1_{\hat 2} \otimes a_{\hat 2}1_{\hat 3} \overset{\text{Axiom }(b)}{=}$
$\widehat{\varepsilon}(1_{\hat 1}a')a_{\hat 1}1_{\hat 1'}1_{\hat 2} \otimes a_{\hat 2}1_{\hat 2'} \overset{\text{Axiom }(a)}{=} \widehat{\varepsilon}(1_{\hat 1}a')(a1)_{\hat 1}1_{\hat 2} \otimes (a1)_{\hat 2} = a_{\hat 1}\widehat{\varepsilon}(1_{\hat 1}a')1_{\hat 2} \otimes a_{\hat 2} = a_{\hat 1}\varepsilon(a') \otimes a_{\hat 2}$
and the other equality follows symmetrically by the other condition in Axiom (b). For the final claim compare both equalities at $a = 1$.

(e) $\varepsilon(\varepsilon(a)a') = \widehat{\varepsilon}(1_{\hat 1}\varepsilon(a)a')1_{\hat 2} \overset{(b)}{=} \widehat{\varepsilon}(1_{\hat 1}\varepsilon(a)\varepsilon(a'))1_{\hat 2} \overset{(d)}{=} \widehat{\varepsilon}(\varepsilon(a)_{\hat 1}\varepsilon(a'))\varepsilon(a)_{\hat 2} \overset{(d)}{=}$
$\widehat{\varepsilon}([\varepsilon(a)\varepsilon(a')]_{\hat 1})[\varepsilon(a)\varepsilon(a')]_{\hat 2} = \varepsilon(a)\varepsilon(a')$

(f) $\varepsilon(1_{\hat 1})\widehat{\varepsilon}(1_{\hat 2}a) = \widehat{\varepsilon}(1_{\hat 2}a)\widehat{\varepsilon}(1_{\hat 1'}1_{\hat 1})1_{\hat 2'} \overset{\text{Axiom }(c)}{=} \widehat{\varepsilon}(1_{\hat 1'}a)1_{\hat 2'} = \varepsilon(a)$

(g) $\varepsilon(a_{\hat 1}) \otimes a_{\hat 2} \overset{(f)}{=} \widehat{\varepsilon}(1_{\hat 2}a_{\hat 1})\varepsilon(1_{\hat 1}) \otimes a_{\hat 2} \overset{(a)}{=} \widehat{\varepsilon}(\varepsilon(1_{\hat 2})a_{\hat 1})\varepsilon(1_{\hat 1}) \otimes a_{\hat 2} \overset{(d)}{=}$
$\varepsilon(1_{\hat 1}) \otimes \widehat{\varepsilon}([\varepsilon(1_{\hat 2})a]_{\hat 1})[\varepsilon(1_{\hat 2})a]_{\hat 2} = \varepsilon(1_{\hat 1}) \otimes \varepsilon(1_{\hat 2})a \overset{(a)}{=} \varepsilon(1_{\hat 1}) \otimes 1_{\hat 2}a$

(h) $\varepsilon(a)\varepsilon(1_{\hat 1}) \otimes 1_{\hat 2} \overset{(e)}{=} \varepsilon(\varepsilon(a)1_{\hat 1}) \otimes 1_{\hat 2} \overset{(d)}{=} \varepsilon(\varepsilon(a)_{\hat 1}) \otimes \varepsilon(a)_{\hat 2} \overset{(g)}{=} \varepsilon(1_{\hat 1}) \otimes 1_{\hat 2}\varepsilon(a)$

(i) $\varepsilon(a_{\hat 1})a_{\hat 2} = \widehat{\varepsilon}(1_{\hat 1}a_{\hat 1})1_{\hat 2}a_{\hat 2} \overset{\text{Axiom }(a)}{=} \widehat{\varepsilon}((1a)_{\hat 1})(1a)_{\hat 2} = \widehat{\varepsilon}(a_{\hat 1})a_{\hat 2} = a$

*Proof of Theorem & Definition 4.5.* Assume first that we are given, as in part (i), a monoidal structure on $\mathsf{mod}(A)$ — with some monoidal product $\square$ and monoidal unit $B$ — and a separable Frobenius structure on $U$ — with monoidal structure $(P^0, P^2)$ and opmonoidal structure $(I^0, I^2)$. Then evaluating the commutative diagrams of Definition 2.5 at the object $X = Y = Z = B$, we see that $UB$ is an algebra with multiplication and unit

$$UB \otimes UB \xrightarrow{P^2_{B,B}} U(B\square B) \xrightarrow{\cong} UB \qquad k \xrightarrow{P^0} UB, \tag{4.4}$$

and a coalgebra with comultiplication and counit

$$UB \xrightarrow{\cong} U(B\square B) \xrightarrow{I^2_{B,B}} UB \otimes UB \qquad UB \xrightarrow{I^0} k. \tag{4.5}$$

The comultiplication is a section of the multiplication since $I^2$ is a section of $P^2$ and commutativity of the diagrams of Definition 4.1 (for the object $X = Y = Z = B$) shows its $B$-bilinearity. That is to say, $UB$ is a separable Frobenius algebra; it should be the base algebra of the desired bialgebroid in part (ii).

Observe that any $A$-module $V$ carries a $UB$-bimodule structure with action occurring in either path of the commutative diagram

$$\begin{tikzcd}
UB \otimes UV \otimes UB \arrow[r, "P^2_{B,V}\otimes 1"] \arrow[d, "1\otimes P^2_{B,V}"'] & U(B\square V) \otimes UB \arrow[r, "\cong"] & UV \otimes UB \arrow[r, "P^2_{V,B}"] & U(V\square B) \arrow[d, "\cong"] \\
UB \otimes U(V\square B) \arrow[r, "\cong"'] & UB \otimes UV \arrow[r, "P^2_{B,V}"'] & U(B\square V) \arrow[r, "\cong"'] & UV
\end{tikzcd} \tag{4.6}$$

and — by the naturality of $P^2$ — any $A$-module map becomes a $UB$-bimodule map for this action. That is to say, $U : \mathsf{mod}(A) \to \mathsf{vec}$ factorizes through a functor $F : \mathsf{mod}(A) \to \mathsf{bim}(UB)$ via the forgetful functor $u : \mathsf{bim}(UB) \to \mathsf{vec}$.

By its associativity, the multiplication of $A$ is an $A$-module map. Then it is a $UB$-bimodule map in the sense that for all $a, a' \in A$ and $b, c \in B$ the equality $(b \cdot a \cdot c)a' = b \cdot aa' \cdot c$ holds. Then in particular

$$(b \cdot 1 \cdot c)(b' \cdot 1 \cdot c') = b \cdot (b' \cdot 1 \cdot c') \cdot c = bb' \cdot 1 \cdot c'c$$

for the unit element 1 of the algebra $A$ and any elements $b,c,b',c'$ of $UB$. That is to say,

$$(UB)^{\mathsf{e}} \to A, \qquad b\otimes c \mapsto b\cdot 1\cdot c$$

is an algebra homomorphism rendering $A$ with the structure of a $(UB)^{\mathsf{e}}$-ring. By Proposition 3.5 there is a corresponding monad $A\boxtimes - : \mathsf{bim}(UB) \to \mathsf{bim}(UB)$ whose forgetful functor is $F : \mathsf{mod}(A) \to \mathsf{bim}(UB)$.

Let us show that $F$ is strict monoidal. Preservation of the monoidal unit amounts to the observation that $FB$ is the $UB$-bimodule living on the vector space $u(FB) = UB$ with both $B$-actions given by the multiplication in $B$: evaluate the diagram of (4.6) at $V = B$. Concerning the monoidal product of any $A$-modules $V$ and $W$, by the separable Frobenius property of $U$, the vector space $u(F(V\square W)) = U(V\square W)$ can be identified with the image of the idempotent map

$$UV\otimes UW \xrightarrow{P^2_{V,W}} U(V\square W) \xrightarrow{I^2_{V,W}} UV\otimes UW,$$

while the vector space $u(FV\otimes_{UB} FW)$ can be identified with the image of the idempotent map

$$UV\otimes UW \to UV\otimes UW, \qquad v\otimes w \mapsto v\cdot 1_{\langle 1\rangle}\otimes 1_{\langle 2\rangle}\cdot w$$

where $1_{\langle 1\rangle}\otimes 1_{\langle 2\rangle}$ denotes the image of the unit element of the algebra $UB$ under the comultiplication (4.5) (with implicit summation understood). So it is enough to compare these idempotent maps. Their equality follows by the commutativity of the following diagram, whose top-right path is the map $-\cdot 1_{\langle 1\rangle}\otimes 1_{\langle 2\rangle}\cdot - : UV\otimes UW \to UV\otimes UW$.

$$\begin{tikzcd}
UV\otimes UW \arrow[r, "1\otimes P^0\otimes 1"] \arrow[dr, "\cong"'] \arrow[dddd, "P^2_{V,W}"'] & UV\otimes UB\otimes UW \arrow[r, "\cong"] \arrow[d, "1\otimes P^2_{B,W}"] \arrow[dl, phantom, "\text{Definition 2.5}"] & UV\otimes U(B\square B)\otimes UW \arrow[r, "1\otimes I^2_{B,B}\otimes 1"] \arrow[d, "1\otimes P^2_{B\square B,W}"'] \arrow[dr, phantom, "\text{Definition 4.1}"] & UV\otimes UB\otimes UB\otimes UW \arrow[d, "1\otimes 1\otimes P^2_{B,W}"] \\
& UV\otimes U(B\square W) \arrow[r, "\cong"'] \arrow[d, equal] & UV\otimes U(B\square B\square W) \arrow[r, "1\otimes I^2_{B,B\square W}"'] & UV\otimes UB\otimes U(B\square W) \arrow[d, "\cong"] \\
& UV\otimes U(B\square W) \arrow[rr, "1\otimes I^2_{B,W}"] \arrow[d, "P^2_{V,B\square W}"] \arrow[drr, phantom, "\text{Definition 4.1}"] & & UV\otimes UB\otimes UW \arrow[d, "P^2_{V,B}\otimes 1"] \\
& U(V\square B\square W) \arrow[rr, "I^2_{V\square B,W}"'] & & U(V\square B)\otimes UW \arrow[d, "\cong"] \\
U(V\square W) \arrow[ur, "\cong"] \arrow[rrr, "I^2_{V,W}"'] & & & UV\otimes UW
\end{tikzcd}$$

The undecorated regions commute by naturality. Since the left $UB$-actions both on $F(V\square W)$ and $FV\otimes_{UB} FW$ are given by the left action on $FV$; and the right $UB$-actions both on $F(V\square W)$ and $FV\otimes_{UB} FW$ are given by the right action on $FW$, we conclude on the equality $F(V\square W) = FV\otimes_{UB} FW$.

We constructed so far a strict monoidal forgetful functor $F$ from the Eilenberg–Moore category of the monad $A\boxtimes - : \mathsf{bim}(UB) \to \mathsf{bim}(UB)$ which — by Theorem & Definition 3.9 — corresponds bijectively to a $UB$-bialgebroid structure on $A$.

Conversely, assume that we are given a bialgebroid structure on $A$ — over some base algebra $B$ — and a separable Frobenius algebra structure on $B$; as in part (ii). These data correspond bijectively to a monoidal structure on $\mathsf{mod}(A)$ such that the forgetful functor $F : \mathsf{mod}(A) \to \mathsf{bim}(B)$ is strict monoidal (see Theorem & Definition 3.9) and a separable Frobenius structure — with monoidal part $(p^0, p^2)$ and monoidal part $(i^0, i^2)$ — on the forgetful functor $u : \mathsf{bim}(B) \to \mathsf{vec}$ (see Proposition 4.4). Hence their composite $uF = U$ admits a separable Frobenius structure (see Exercise 4.3) with monoidal part

$$u(FV) \otimes u(FW) \xrightarrow{p^2_{FV,FW}} u(FV \otimes_B FW) = uF(V \otimes_B W) = U(V \otimes_B W)$$

$$k \xrightarrow{\qquad p^0 \qquad} uB = u(FB) = UB$$

and opmonoidal part

$$U(V \otimes_B W) = uF(V \otimes_B W) = u(FV \otimes_B FW) \xrightarrow{i^2_{FV,FW}} u(FV) \otimes u(FW)$$

$$UB = u(FB) = uB \xrightarrow{\qquad i^0 \qquad} k$$

for any $A$-modules $V$ and $W$.

These correspondences above, between the data in parts (i) and (ii), are clearly mutual inverses.

Assume again that we are given a bialgebroid structure on $A$ — with $B^{\mathsf{e}}$-ring structure $\eta : B^{\mathsf{e}} \to A$ — and a separable Frobenius algebra structure on its base algebra $B$; as in part (ii). Once again, we want to use that the separable Frobenius algebra structure on $B$ is equivalently a separable Frobenius structure $(p^0, p^2, i^0, i^2)$ on the forgetful functor $\mathsf{bim}(B) \to \mathsf{vec}$. At the $B$-bimodule $|A$ of (3.3) the binary part of the opmonoidal structure takes the explicit form

$$i^2_{|A,|A}(a' \otimes_B a) = \eta(1 \otimes 1_{\langle 1\rangle})a' \otimes \eta(1_{\langle 2\rangle} \otimes 1)a$$

in terms of the image $1_{\langle 1\rangle} \otimes 1_{\langle 2\rangle}$ of the unit element of $B$ under the comultiplication $i^2_{B,B}$ (where implicit summation is understood). The stated coalgebra structure on $A$ is constructed from the comultiplication $\Delta : |A \to |A \otimes_B |A$ and the counit $\varepsilon : |A \to B$ of the $B$-bialgebroid $A$ as

$$\widehat{\Delta} :\ A \xrightarrow{\Delta} |A \otimes_B |A \xrightarrow{i^2_{|A,|A}} A \otimes A \qquad \widehat{\varepsilon} :\ A \xrightarrow{\varepsilon} B \xrightarrow{i^0} k.$$

Its coassociativity and counitality are immediate from the coassociativity and counitality of the bialgebroid $A$ (axioms (a) and (b) of Definition 3.6) and commutativity of the diagrams of Definition 2.5 for $(i^0, i^2)$.

There are two comultiplications present: $\Delta$ for the $B|B$-coring $A$ and $\widehat{\Delta}$ for the coalgebra $A$. We use two variants of Sweedler-Heyneman's implicit summation index notation for them. We write $\Delta(a) = a_1 \otimes a_2$ and $\widehat{\Delta}(a) = a_{\hat{1}} \otimes a_{\hat{2}}$ for any $a \in A$.

Let us turn to checking the compatibility conditions between the algebra and coalgebra structures of $A$. Identity (a) of part (iii) holds for any $a, a' \in A$ by

$$\begin{aligned}
\widehat{\Delta}(a)\widehat{\Delta}(a') &= \eta(1 \otimes 1_{\langle 1\rangle})a_1\eta(1 \otimes 1_{\langle 1'\rangle})a'_1 \otimes \eta(1_{\langle 2\rangle} \otimes 1)a_2\eta(1_{\langle 2'\rangle} \otimes 1)a'_2 \\
&= \eta(1 \otimes 1_{\langle 1\rangle})a_1a'_1 \otimes \eta(1_{\langle 2\rangle} \otimes 1)a_2\eta(1_{\langle 1'\rangle} \otimes 1)\eta(1_{\langle 2'\rangle} \otimes 1)a'_2 \\
&= \eta(1 \otimes 1_{\langle 1\rangle})a_1a'_1 \otimes \eta(1_{\langle 2\rangle} \otimes 1)a_2a'_2 \\
&= \eta(1 \otimes 1_{\langle 1\rangle})(aa')_1 \otimes \eta(1_{\langle 2\rangle} \otimes 1)(aa')_2 = \widehat{\Delta}(aa').
\end{aligned}$$

The second equality follows by axiom (d) in Definition 3.6; in the third equality we used that

$$\eta(1_{\langle 1'\rangle}\otimes 1)\eta(1_{\langle 2'\rangle}\otimes 1)=\eta(1_{\langle 1'\rangle}1_{\langle 2'\rangle}\otimes 1)=\eta(1\otimes 1)=1\otimes 1$$

by the fact that the comultiplication of $B$ is a section of the multiplication; and in the penultimate equality we used the multiplicativity of $\Delta$; that is, condition (a) in part (ii) of Theorem & Definition 3.9.

The comultiplication $\Delta$ of the bialgebroid $A$ is unital by condition (b) in Theorem & Definition 3.9 from which it follows that

$$\widehat{\Delta}(1)=\eta(1\otimes 1_{\langle 1\rangle})\otimes\eta(1_{\langle 2\rangle}\otimes 1).$$

Then

$$\begin{aligned}(\widehat{\Delta}(1)\otimes 1)(1\otimes\widehat{\Delta}(1)) &= \eta(1\otimes 1_{\langle 1\rangle})\otimes\eta(1_{\langle 2\rangle}\otimes 1)\eta(1\otimes 1_{\langle 1'\rangle})\otimes\eta(1_{\langle 2'\rangle}\otimes 1)\\ &= \eta(1\otimes 1_{\langle 1\rangle})\otimes\eta(1_{\langle 2\rangle}\otimes 1_{\langle 1'\rangle})\otimes\eta(1_{\langle 2'\rangle}\otimes 1)\\ &= \eta(1\otimes 1_{\langle 1\rangle})\otimes\eta(1\otimes 1_{\langle 1'\rangle})\eta(1_{\langle 2\rangle}\otimes 1)\otimes\eta(1_{\langle 2'\rangle}\otimes 1)\\ &= (1\otimes\widehat{\Delta}(1))(\widehat{\Delta}(1)\otimes 1)\end{aligned}$$

and since $\Delta$ is a $B$-bimodule map this is equal also to $1_{\hat{1}}\otimes 1_{\hat{2}}\otimes 1_{\hat{3}}$, proving that axiom (b) of part (iii) holds.

Finally, since $\Delta$ is a right $B^{\mathrm{e}}$-module map in the sense of axiom (c) in Definition 3.6, it follows that

$$\begin{aligned}\widehat{\Delta}(a\eta(b\otimes b')) &= i^2_{|A,|A}(\Delta(a\eta(b\otimes b')))\\ &= i^2_{|A,|A}(a_1\eta(b\otimes 1)\otimes a_2\eta(1\otimes b'))=a_{\hat{1}}\eta(b\otimes 1)\otimes a_{\hat{2}}\eta(1\otimes b')\end{aligned}$$

for any $a$ and $b,b'\in B$. On the other hand, by condition (c) in part (ii) of Theorem & Definition 3.9,

$$\begin{aligned}&\widehat{\varepsilon}(aa')=i^0(\varepsilon(aa'))=i^0(\varepsilon(a\eta(1\otimes\varepsilon(a'))))=\widehat{\varepsilon}(a\eta(1\otimes\varepsilon(a'))) \qquad \text{and}\\ &\widehat{\varepsilon}(aa')=i^0(\varepsilon(aa'))=i^0(\varepsilon(a\eta(\varepsilon(a')\otimes 1)))=\widehat{\varepsilon}(a\eta(\varepsilon(a')\otimes 1)).\end{aligned}$$

Using these identities together with the fact that $\widehat{\varepsilon}$ is the counit of $\widehat{\Delta}$, it follows for any $a,a',a''\in A$ that

$$\begin{aligned}\widehat{\varepsilon}(aa'_{\hat{1}})\widehat{\varepsilon}(a'_{\hat{2}}a'') &= \widehat{\varepsilon}(aa'_{\hat{1}})\widehat{\varepsilon}(a'_{\hat{2}}\eta(1\otimes\varepsilon(a'')))\\ &= \widehat{\varepsilon}(a[a'\eta(1\otimes\varepsilon(a''))]_{\hat{1}}\widehat{\varepsilon}([a'\eta(1\otimes\varepsilon(a''))]_{\hat{2}}))\\ &= \widehat{\varepsilon}(aa'\eta(1\otimes\varepsilon(a'')))=\widehat{\varepsilon}(aa'a'') \qquad \text{and}\\ \widehat{\varepsilon}(aa'_{\hat{2}})\widehat{\varepsilon}(a'_{\hat{1}}a'') &= \widehat{\varepsilon}(aa'_{\hat{2}})\widehat{\varepsilon}(a'_{\hat{1}}\eta(\varepsilon(a'')\otimes 1))\\ &= \widehat{\varepsilon}(a[a'\eta(\varepsilon(a'')\otimes 1)]_{\hat{2}}\widehat{\varepsilon}([a'\eta(\varepsilon(a'')\otimes 1)]_{\hat{1}}))\\ &= \widehat{\varepsilon}(aa'\eta(\varepsilon(a'')\otimes 1))=\widehat{\varepsilon}(aa'a'')\end{aligned}$$

so that also axiom (c) of part (iii) holds.

Conversely, assume that a coalgebra structure $(\widehat{\varepsilon},\widehat{\Delta})$ of $A$ as in part (iii) is given. First we construct the separable Frobenius algebra $B$ which should be the base algebra of the bialgebroid in part (ii) to be constructed next.

By part (c) of Exercise 4.6, the map $\varepsilon : A\to A$ of (4.3) is idempotent. As a vector space, let $B$ be its image $\varepsilon(A)$. By parts (a) and (e) of Exercise 4.6 it is a unital subalgebra of $A$.

By part (b) of Exercise 4.6 we can define a counit

$$B \to k, \qquad \varepsilon(a) \mapsto \widehat{\varepsilon}\varepsilon(a) = \widehat{\varepsilon}(a)$$

and by parts (a) and (h) of Exercise 4.6 we can define a $B$-bilinear comultiplication

$$B \to B \otimes B, \qquad \varepsilon(a) \mapsto \varepsilon(a)\varepsilon(1_{\hat{1}}) \otimes 1_{\hat{2}} = \varepsilon(1_{\hat{1}}) \otimes 1_{\hat{2}}\varepsilon(a).$$

It is coassociative by part (h), and counital by parts (a) and (b) of Exercise 4.6. It is a section of the multiplication by part (i) of Exercise 4.6. Thus we equipped $B$ with the structure of a separable Frobenius algebra.

Symmetrically to (4.3) consider the map

$$\overline{\varepsilon} : A \to A, \qquad a \mapsto 1_{\hat{1}}\widehat{\varepsilon}(1_{\hat{2}}a).$$

By symmetric considerations to those in Exercise 4.6 it is an idempotent map whose image is a unital subalgebra of $A$. By part (b) of Exercise 4.6, $\overline{\varepsilon} \circ \varepsilon = \overline{\varepsilon}$ and symmetrically, $\varepsilon \circ \overline{\varepsilon} = \varepsilon$. Hence they restrict to mutually inverse isomorphisms between the vector spaces $B = \varepsilon(A)$ and $\overline{\varepsilon}(A)$. Using Axiom (c) of part (iii) in the second equality, part (d) of Exercise 4.6 in the third one, and its part (b) in the fourth one,

$$\begin{aligned}
\overline{\varepsilon}(\varepsilon(a)a') &= 1_{\hat{1}}\widehat{\varepsilon}(1_{\hat{2}}\varepsilon(a)a') = 1_{\hat{1}}\widehat{\varepsilon}(1_{\hat{2}}\varepsilon(a)_{\hat{1}})\widehat{\varepsilon}(\varepsilon(a)_{\hat{2}}a') \\
&= 1_{\hat{1}}\widehat{\varepsilon}(1_{\hat{2}}1_{\hat{1}'}\varepsilon(a))\widehat{\varepsilon}(1_{\hat{2}'}a') = 1_{\hat{1}}\widehat{\varepsilon}(1_{\hat{2}}1_{\hat{1}'}a)\widehat{\varepsilon}(1_{\hat{2}'}a') = \overline{\varepsilon}(\overline{\varepsilon}(a')a)
\end{aligned} \tag{4.7}$$

for any $a, a' \in A$. Symmetrically to part (e) of Exercise 4.6, $\overline{\varepsilon}(\overline{\varepsilon}(a')a) = \overline{\varepsilon}(a')\overline{\varepsilon}(a)$. Using this identity we infer from (4.7) that

$$\overline{\varepsilon}(\varepsilon(a)\varepsilon(a')) = \overline{\varepsilon}\varepsilon(a')\overline{\varepsilon}(a) = \overline{\varepsilon}\varepsilon(a')\overline{\varepsilon}\varepsilon(a).$$

This proves that the subalgebras $B = \varepsilon(A)$ and $\overline{\varepsilon}(A)$ of $A$ are anti-isomorphic. Moreover, the elements of $\overline{\varepsilon}(A)$ commute with the elements of $B$ by part (d) of Exercise 4.6. Hence there is an algebra homomorphism

$$B \otimes B^{\mathsf{op}} \to A, \qquad \varepsilon(a) \otimes \varepsilon(a') \mapsto \varepsilon(a)\overline{\varepsilon}(a') = \overline{\varepsilon}(a')\varepsilon(a)$$

rendering $A$ with the structure of a $B^{\mathsf{e}}$-ring.

The candidate comultiplication of the $B$-bialgebroid $A$ is the composite map

$$\Delta : \; A \xrightarrow{\widehat{\Delta}} A \otimes A \twoheadrightarrow {}_{|}A \otimes_B {}_{|}A \; .$$

By the second condition in part (d) of Exercise 4.6 and its symmetric counterpart

$$\widehat{\Delta}(\overline{\varepsilon}(a)a') = a'_{\hat{1}} \otimes \overline{\varepsilon}(a)a'_{\hat{2}} \tag{4.8}$$

this is a $B$-bimodule map ${}_{|}A \to {}_{|}A \otimes_B {}_{|}A$. The candidate counit is $\varepsilon : A \to B$ in (4.3). Symmetrically to (4.7), $\varepsilon(\overline{\varepsilon}(a')a) = \varepsilon(a)\varepsilon(a')$. Together with part (e) of Exercise 4.6 this proves the bilinearity of $\varepsilon : {}_{|}A \to B$. The comultiplication $\Delta$ is coassociative by the coassociativity of $\widehat{\Delta}$ and counital by part (i) of Exercise 4.6 and its symmetric counterpart

$$\overline{\varepsilon}(a_{\hat{2}})a_{\hat{1}} = a. \tag{4.9}$$

Symmetrically to part (a) of Exercise 4.6, $\overline{\varepsilon}(1_{\hat{1}}) \otimes 1_{\hat{2}} = 1_{\hat{1}} \otimes 1_{\hat{2}}$. Thus applying $\overline{\varepsilon}$ to the first tensor factor of part (h) of Exercise 4.6, we get the identity

$$1_{\hat{1}}\overline{\varepsilon}(a) \otimes 1_{\hat{2}} = 1_{\hat{1}} \otimes 1_{\hat{2}}\varepsilon(a). \tag{4.10}$$

Together with the multiplicativity of $\widehat{\Delta}$ this implies axiom (d) in Definition 3.6. By the multiplicativity of $\widehat{\Delta}$ also $\Delta$ is multiplicative and it is unital by construction; that is, conditions

(a) and (b) in part (ii) of Theorem & Definition 3.9 hold. Then from its left $B^{\mathrm{e}}$-linearity also the right $B^{\mathrm{e}}$-linearity of $\Delta$ in the sense of axiom (c) in Definition 3.6 follows. The counit $\varepsilon$ preserves the unit by part (a) of Exercise 4.6 so that condition (d) in Theorem & Definition 3.9 holds. Finally, condition (e) in Definition 3.6 and condition (c) in Theorem & Definition 3.9 follow by part (c) of Exercise 4.6 and its symmetric counterpart. Summarizing, we constructed a $B$-bialgebroid $A$.

For any $B$-bialgebroid $A$, the map

$$B \to A, \qquad b \mapsto \eta(b \otimes 1) \tag{4.11}$$

is a monomorphism split by the counit $\varepsilon$. Hence we may identify the base algebra $B$ with its isomorphic image in $A$ under (4.11). Up-to this identification, the above constructions between the data in parts (ii) and (iii) are clearly mutual inverses.

It remains to prove the final claim about the antipode. Assume first that the bialgebroid in part (ii) is a Hopf algebroid; that is, the map

$$\beta : A\overline{\varepsilon}(1_{\hat{2}}) \otimes 1_{\hat{1}}A \cong A \otimes_{B^{\mathrm{op}}} A \to A \otimes_B A \cong 1_{\hat{1}}A \otimes 1_{\hat{2}}A, \quad a\overline{\varepsilon}(1_{\hat{2}}) \otimes 1_{\hat{1}}a' \mapsto a_{\hat{1}} \otimes a_{\hat{2}}a' \tag{4.12}$$

is invertible. Then we claim that the antipode $\sigma$ is constructed as the composite map

$$\sigma : \ A \xrightarrow{1_{\hat{1}} - \otimes 1_{\hat{2}}} 1_{\hat{1}}A \otimes 1_{\hat{2}}A \xrightarrow{\beta^{-1}} A\overline{\varepsilon}(1_{\hat{2}}) \otimes 1_{\hat{1}}A \xrightarrow{\widehat{\varepsilon} \otimes 1} A.$$

In order to see that it satisfies the antipode axioms indeed, note that the map $\beta$ of (4.12) renders commutative the following diagrams.

$$\begin{tikzcd}
A\overline{\varepsilon}(1_{\hat{2}}) \otimes 1_{\hat{1}}A \otimes A \arrow[r, "\beta\otimes 1"] \arrow[d, "1\otimes m"'] & 1_{\hat{1}}A \otimes 1_{\hat{2}}A \otimes A \arrow[d, "1\otimes m"] \\
A\overline{\varepsilon}(1_{\hat{2}}) \otimes 1_{\hat{1}}A \arrow[r, "\beta"'] & 1_{\hat{1}}A \otimes 1_{\hat{2}}A
\end{tikzcd}
\qquad
\begin{tikzcd}
A\overline{\varepsilon}(1_{\hat{2}}) \otimes 1_{\hat{1}}A \arrow[r, "\beta"] \arrow[d, "\widehat{\Delta}\otimes 1"'] & 1_{\hat{1}}A \otimes 1_{\hat{2}}A \arrow[d, "\widehat{\Delta}\otimes 1"] \\
A \otimes A\overline{\varepsilon}(1_{\hat{2}}) \otimes 1_{\hat{1}}A \arrow[r, "1\otimes\beta"'] & A \otimes 1_{\hat{1}}A \otimes 1_{\hat{2}}A
\end{tikzcd}$$

$$\begin{tikzcd}
A\overline{\varepsilon}(1_{\hat{2}}) \otimes 1_{\hat{1}}A \arrow[r, "\beta"] \arrow[dr, "m"'] & 1_{\hat{1}}A \otimes 1_{\hat{2}}A \arrow[d, "\widehat{\varepsilon}\otimes 1"] \\
& A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "-\overline{\varepsilon}(1_{\hat{2}})\otimes 1_{\hat{1}}"] \arrow[dr, "\widehat{\Delta}"'] & A\overline{\varepsilon}(1_{\hat{2}}) \otimes 1_{\hat{1}}A \arrow[d, "\beta"] \\
& 1_{\hat{1}}A \otimes 1_{\hat{2}}A
\end{tikzcd}$$

Making use of them, the first two antipode axioms in the claim are checked by the same steps as in the bialgebra case in the proof of Theorem & Example 3.3. In order to verify the third antipode axiom, note that it follows from (4.10) and the right $A$-linearity of $\beta$ that for any $a, a' \in A$

$$\begin{aligned}
\sigma(\overline{\varepsilon}(a)a') &= (\widehat{\varepsilon} \otimes 1)(\beta^{-1}(1_{\hat{1}}\overline{\varepsilon}(a)a' \otimes 1_{\hat{2}})) = (\widehat{\varepsilon} \otimes 1)(\beta^{-1}(1_{\hat{1}}a' \otimes 1_{\hat{2}}\varepsilon(a))) \\
&= (\widehat{\varepsilon} \otimes 1)(\beta^{-1}(1_{\hat{1}}a' \otimes 1_{\hat{2}}))\varepsilon(a) = \sigma(a')\varepsilon(a).
\end{aligned}$$

From this, and the counitality condition (4.9) we conclude that

$$\sigma(a_{\hat{1}})a_{\hat{2}}\sigma(a_{\hat{3}}) = \sigma(a_{\hat{1}})\varepsilon(a_{\hat{2}}) = \sigma(\overline{\varepsilon}(a_{\hat{2}})a_{\hat{1}}) = \sigma(a)$$

as needed.

Conversely, assume that the weak bialgebra in part (iii) admits an antipode $\sigma$. In terms of it, consider the map

$$A\otimes A\to A\otimes A, \qquad a\otimes a'\mapsto a_{\hat{1}}\otimes \sigma(a_{\hat{2}})a'. \tag{4.13}$$

In order to see that it restricts to a map $1_{\hat{1}}A\otimes 1_{\hat{2}}A\to A\overline{\varepsilon}(1_{\hat{2}})\otimes 1_{\hat{1}}A$, note that by the third and the first antipode axioms $\sigma(a)=\sigma(a_{\hat{1}})\varepsilon(a_{\hat{2}})$ for all $a\in A$. Using this in the first and the last equalities, together with (4.10) in the second equality and (4.8) in the third one, for any $a,a'\in A$ we have

$$\begin{aligned}\sigma(a)\varepsilon(a') &= \sigma(a_{\hat{1}})\widehat{\varepsilon}(1_{\hat{1}}a_{\hat{2}})1_{\hat{2}}\varepsilon(a')=\sigma(a_{\hat{1}})\widehat{\varepsilon}(1_{\hat{1}}\overline{\varepsilon}(a')a_{\hat{2}})1_{\hat{2}}\\ &= \sigma([\overline{\varepsilon}(a')a]_{\hat{1}})\widehat{\varepsilon}(1_{\hat{1}}[\overline{\varepsilon}(a')a]_{\hat{2}})1_{\hat{2}}=\sigma(\overline{\varepsilon}(a')a).\end{aligned} \tag{4.14}$$

Symmetrically,

$$[1_{\hat{1}}\widehat{\varepsilon}(a'1_{\hat{2}})]\sigma(a)=\sigma(a[\widehat{\varepsilon}(a'1_{\hat{1}})1_{\hat{2}}]). \tag{4.15}$$

After these preparations we see that (4.13) sends $1_{\hat{1}}a\otimes 1_{\hat{2}}a'$ to

$$\begin{aligned}a_{\hat{1}}\otimes\sigma(1_{\hat{1}}a_{\hat{2}})1_{\hat{2}}a' &= a_{\hat{1}}\otimes\sigma(a_{\hat{2}})\varepsilon(1_{\hat{1}})1_{\hat{2}}a'=a_{\hat{1}}\otimes\sigma(a_{\hat{2}})a'\\ &= a_{\hat{1}}1_{\hat{1}}\otimes\sigma(a_{\hat{2}}\widehat{\varepsilon}(1_{\hat{2}})1_{\hat{3}})a'=a_{\hat{1}}1_{\hat{1}}\otimes\sigma(a_{\hat{2}}\widehat{\varepsilon}(1_{\hat{2}}1_{\hat{1}'})1_{\hat{2}'})a'\\ &= a_{\hat{1}}1_{\hat{1}}\otimes 1_{\hat{1}'}\widehat{\varepsilon}(1_{\hat{2}}1_{\hat{2}'})\sigma(a_{\hat{2}})a'=a_{\hat{1}}\overline{\varepsilon}(1_{\hat{2}'})\otimes 1_{\hat{1}'}\sigma(a_{\hat{2}})a'.\end{aligned}$$

First we used (4.8), then (4.14), the counitality condition in part (i) of Exercise 4.6, the multiplicativity and the counitality of $\widehat{\Delta}$, Axiom (b) of weak bialgebras and (4.15).

In order to see that the so obtained map

$$\beta^{-1}:1_{\hat{1}}A\otimes 1_{\hat{2}}A\to A\overline{\varepsilon}(1_{\hat{2}})\otimes 1_{\hat{1}}A, \qquad 1_{\hat{1}}a\otimes 1_{\hat{2}}a'\mapsto a_{\hat{1}}\otimes\sigma(a_{\hat{2}})a'$$

is indeed the inverse of $\beta$, observe that by (4.8),

$$a_{\hat{1}}\otimes\varepsilon(a_{\hat{2}})=a_{\hat{1}}\otimes\widehat{\varepsilon}(1_{\hat{1}}a_{\hat{2}})1_{\hat{2}}=[1_{\hat{1}}a]_{\hat{1}}\widehat{\varepsilon}([1_{\hat{1}}a]_{\hat{2}})\otimes 1_{\hat{2}}=1_{\hat{1}}a\otimes 1_{\hat{2}}$$

and, similarly, $a_{\hat{1}}\otimes 1_{\hat{1}}\widehat{\varepsilon}(a_{\hat{2}}1_{\hat{2}})=a\overline{\varepsilon}(1_{\hat{2}})\otimes 1_{\hat{1}}$ for all $a\in A$. With these identities at hand it follows by the first and the second antipode axioms, respectively, that

$$\begin{aligned}\beta\beta^{-1}(1_{\hat{1}}a\otimes 1_{\hat{2}}a') &= a_{\hat{1}}\otimes a_{\hat{2}}\sigma(a_{\hat{3}})a'=a_{\hat{1}}\otimes\varepsilon(a_{\hat{2}})a'=1_{\hat{1}}a\otimes 1_{\hat{2}}a' \quad \text{and}\\ \beta^{-1}\beta(a\overline{\varepsilon}(1_{\hat{2}})\otimes 1_{\hat{1}}a') &= a_{\hat{1}}\otimes\sigma(a_{\hat{2}})a_{\hat{3}}a'=a_{\hat{1}}\otimes 1_{\hat{1}}\widehat{\varepsilon}(a_{\hat{2}}1_{\hat{2}})a'=a\overline{\varepsilon}(1_{\hat{2}})\otimes 1_{\hat{1}}a'.\end{aligned}$$

□

**Corollary 4.7.** *The category* $\mathsf{mod}(A)$ *of a weak bialgebra* $A$ *is monoidal via the module tensor product over the base subalgebra. If* $A$ *is a weak Hopf algebra then* $\mathsf{mod}(A)$ *is right closed as well.*

Note that the axioms of a weak (Hopf) bialgebra are formally self-dual. That is to say, drawing the axioms as commutative diagrams,

$$\begin{CD} A\otimes A @>{\widehat{\Delta}\otimes\widehat{\Delta}}>> A\otimes A\otimes A\otimes A @>{1\otimes c\otimes 1}>> A\otimes A\otimes A\otimes A\\ @V{m}VV @. @VV{m\otimes m}V\\ A @>>{\widehat{\Delta}}> @. A\otimes A \end{CD}$$

$$\begin{tikzcd}
k \arrow[r, "u\otimes u"] \arrow[d, "u\otimes u"'] \arrow[dr, "u"] & A\otimes A \arrow[r, "\widehat{\Delta}\otimes\widehat{\Delta}"] & A\otimes A\otimes A\otimes A \arrow[dd, "1\otimes m\otimes 1"] \\
A\otimes A \arrow[d, "\widehat{\Delta}\otimes\widehat{\Delta}"'] & A \arrow[dr, "(\widehat{\Delta}\otimes 1)\circ\widehat{\Delta}"] & \\
A\otimes A\otimes A\otimes A \arrow[r, "1\otimes c\otimes 1"'] & A\otimes A\otimes A\otimes A \arrow[r, "1\otimes m\otimes 1"'] & A\otimes A\otimes A
\end{tikzcd}$$

$$\begin{tikzcd}
A\otimes A\otimes A \arrow[r, "1\otimes\widehat{\Delta}\otimes 1"] \arrow[dd, "1\otimes\widehat{\Delta}\otimes 1"'] \arrow[dr, "m\circ(m\otimes 1)"] & A\otimes A\otimes A\otimes A \arrow[r, "1\otimes c\otimes 1"] & A\otimes A\otimes A\otimes A \arrow[d, "m\otimes m"] \\
 & A \arrow[dr, "\widehat{\varepsilon}"] & A\otimes A \arrow[d, "\widehat{\varepsilon}\otimes\widehat{\varepsilon}"] \\
A\otimes A\otimes A\otimes A \arrow[r, "m\otimes m"'] & A\otimes A \arrow[r, "\widehat{\varepsilon}\otimes\widehat{\varepsilon}"'] & k
\end{tikzcd}$$

— where $c : A\otimes A \to A\otimes A$ is the flip map $a\otimes a' \mapsto a'\otimes a$ — this set of diagrams is invariant under reversing the arrows and interchanging the roles of the algebra and the coalgebra structures. As an immediate consequence of this symmetry, also the category of comodules over a weak bialgebra is monoidal, and the category of comodules over a weak Hopf algebra is right closed.

**Examples 4.8.**

(1) Let $B$ be a separable Frobenius algebra with counit $i^0$ and comultiplication $b \mapsto b_{\langle 1\rangle}\otimes b_{\langle 2\rangle}$ (where implicit summation is understood). From Example 3.10 and Theorem & Definition 4.5 it follows that there is a weak Hopf algebra $B^e$. Its comultiplication $\widehat{\Delta}$, counit $\widehat{\varepsilon}$ and antipode $\sigma$ take the following forms for all $b\otimes b' \in B^{\mathrm{e}}$.

$$\widehat{\Delta}(b\otimes b') = (b\otimes 1_{\langle 1\rangle})\otimes(1_{\langle 2\rangle}\otimes b') \quad \widehat{\varepsilon}(b\otimes b') = i^0(bb') \quad \sigma(b\otimes b') = b'\otimes 1_{\langle 1\rangle} i^0(b1_{\langle 2\rangle})$$

(2) Consider a category $\mathsf{C}$ which has finitely many objects and whose morphisms constitute a proper set. For any field $k$, take the vector space $k\mathsf{C}$ spanned by the morphisms of $\mathsf{C}$.

First we equip $k\mathsf{C}$ with an algebra structure. Define the product of two morphisms $f$ and $g$ to be $f\circ g$ if they are composable (that is, the source of $f$ and the target of $g$ coincide) and zero otherwise. Extending it linearly to $k\mathsf{C}$ in both arguments, we obtain an associative algebra whose unit is $\sum_{X\in\mathsf{C}^0} 1_X$; the sum of the identity morphisms $1_X$ for all objects $X$ of $\mathsf{C}$ (remember that this is a finite sum by assumption).

Observe that the above algebra $k\mathsf{C}$ admits a weak bialgebra structure with comultiplication $\widehat{\Delta}$ and counit $\widehat{\varepsilon}$ defined on any morphism $f$ of $\mathsf{C}$ as

$$\widehat{\Delta}(f) = f\otimes f \qquad \widehat{\varepsilon}(f) = 1$$

and linearly extended to $k\mathsf{C}$. Let us stress that $\widehat{\Delta}$ takes the unit element $\sum_{X\in\mathsf{C}^0} 1_X$ of $k\mathsf{C}$ to $\sum_{X\in\mathsf{C}^0}(1_X\otimes 1_X)$, which differs from $(\sum_{X\in\mathsf{C}^0} 1_X)\otimes(\sum_{X\in\mathsf{C}^0} 1_X)$ unless $\mathsf{C}$ has only one object. That is to say, in general $\widehat{\Delta}$ does not preserve the unit element.

As in Theorem & Definition 4.5, the above weak bialgebra $k\mathsf{C}$ can be regarded as a bialgebroid over the separable Frobenius algebra $k\mathsf{C}^0$ spanned by the objects of $\mathsf{C}$; with multiplication

$$XY = \begin{cases} X & \text{if } X = Y \\ 0 & \text{if } X \neq Y \end{cases}$$

and comultiplication $X \mapsto X \otimes X$.

If $\mathsf{C}$ is in addition a groupoid — that is, any of its morphisms is invertible — then the above weak bialgebra is a weak Hopf algebra with antipode $\sigma$ defined on any morphism $f$ of $\mathsf{C}$ as $\sigma(f) = f^{-1}$ and linearly extended to $k\mathsf{C}$.

This example extends the well-known fact that the linear spans of monoids are bialgebras and the linear spans of groups are Hopf algebras. In this way it gives an explanation of the origin of the term 'algebroid'.

To begin further reading about weak (Hopf) bialgebras with, we recommend e.g. [10] [12] [25] [24].

## 5. Lecture: (Hopf) bimonoids in duoidal categories

The subject of the fifth lecture is *(Hopf) bimonoids* in so-called duoidal categories. Helpful references should be [1] and [9].

**Definition 5.1.** A *monoid* in a monoidal category $(\mathsf{A}, \otimes, I)$ is a triple consisting of an object $M$, a morphisms $m : M \otimes M \to M$ — called the *multiplication* — a morphism $u : I \to M$ — called the *unit* — for which the following diagrams commute.

$$
\begin{tikzcd}
M \otimes M \otimes M \arrow[r, "m\otimes 1"] \arrow[d, "1\otimes m"'] & M \otimes M \arrow[d, "m"] \\
M \otimes M \arrow[r, "m"'] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[r, "u\otimes 1"] \arrow[d, "1\otimes u"'] \arrow[dr, equal] & M \otimes M \arrow[d, "m"] \\
M \otimes M \arrow[r, "m"'] & M
\end{tikzcd}
$$

A *morphism of monoids* is a morphism $f : M \to M'$ in $\mathsf{A}$ which is compatible with the multiplications and the units in the sense of the following commutative diagrams.

$$
\begin{tikzcd}
M \otimes M \arrow[r, "f\otimes f"] \arrow[d, "m"'] & M' \otimes M' \arrow[d, "m'"] \\
M \arrow[r, "f"'] & M'
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, equal] \arrow[d, "u"'] & I \arrow[d, "u'"] \\
M \arrow[r, "f"'] & M'
\end{tikzcd}
$$

A *comonoid* in a monoidal category is a monoid in the opposite category; that is, a triple $(C, d : C \to C \otimes C, e : C \to I)$ making commutative the same diagrams with reversed arrows. A *morphism of comonoids* is a morphism of monoids in the opposite category; that is, it renders commutative the same diagrams with reversed vertical arrows.

**Examples 5.2.**

(1) A monoid in the monoidal category of sets (see part (1) of Example 2.2) is an ordinary monoid (i.e. a set with an associative multiplication map and unit element).
(2) A monoid in the monoidal category of vector spaces over a given field $k$ (see part (2) of Example 2.2) is a $k$-algebra.
(3) A monoid in the monoidal category of bimodules over some algebra $A$ (see part (3) of Example 2.2) is an $A$-ring.
(4) A monoid in the monoidal category of endofunctors on some category $\mathsf{A}$ (see part (4) of Example 2.2) is a monad on $\mathsf{A}$.

(5) The monoidal unit $I$ of any monoidal category $(\mathsf{A},\otimes,I)$ is a monoid with unit provided by the identity morphism $I \to I$ and multiplication $I \otimes I \cong I$. Symmetrically, it is a comonoid as well with counit equal to the identity morphism $I \to I$ and comultiplication $I \cong I \otimes I$.

(6) If $(f, f^0, f^2)$ is a monoidal functor $(\mathsf{A},\otimes,I) \to (\mathsf{A}',\otimes',I')$ then for any monoid $(M,m,u)$ in $(\mathsf{A},\otimes,I)$ there is a monoid $fM$ in $(\mathsf{A}',\otimes',I')$. The multiplication and the unit are

$$fM \otimes' fM \xrightarrow{f^2_{M,M}} f(M\otimes M) \xrightarrow{fm} fM \quad \text{and} \quad I' \xrightarrow{f^0} fI \xrightarrow{fu} fM.$$

Symmetrically, if $(f, f^0, f^2)$ is an opmonoidal functor then $fC$ inherits the structure of comonoid in $(\mathsf{A}',\otimes',I')$ for any comonoid $C$ in $(\mathsf{A},\otimes,I)$.

**Exercise 5.3.** Show that any monoid $(M,m,u)$ in a monoidal category $(\mathsf{A},\otimes,I)$ induces a monad $M \otimes -$ on $\mathsf{A}$ with multiplication and unit which have the respective components

$$m \otimes 1 : M \otimes M \otimes X \to M \otimes X \quad \text{and} \quad u \otimes 1 : X \to M \otimes X$$

when evaluated at an arbitrary object $X$.

**Definition 5.4.** [1, 31] A *duoidal structure* on a category $\mathsf{A}$ consists of

- *two* monoidal structures $(\diamond, I)$ and $(\blacklozenge, J)$
- a monoidal structure

$$(\xi^0 : J \to J \diamond J,\ \xi_{X,Y,V,Z} : (X \diamond Y) \blacklozenge (V \diamond Z) \to (X \blacklozenge V) \diamond (Y \blacklozenge Z))$$

on the functor $\diamond : (\mathsf{A}, \blacklozenge, J) \times (\mathsf{A}, \blacklozenge, J) \to (\mathsf{A}, \blacklozenge, J)$ and a monoidal structure

$$(\xi^0_0 : J \to I,\ \xi_0 : I \blacklozenge I \to I)$$

on the functor $I : (\mathbb{1},1,1) \to (\mathsf{A}, \blacklozenge, J)$,

*equivalently,* an opmonoidal structure

$$(\xi_0 : I \blacklozenge I \to I,\ \xi_{X,Y,V,Z} : (X \diamond Y) \blacklozenge (V \diamond Z) \to (X \blacklozenge V) \diamond (Y \blacklozenge Z))$$

on the functor $\blacklozenge : (\mathsf{A},\diamond,I) \times (\mathsf{A},\diamond,I) \to (\mathsf{A},\diamond,I)$ and an opmonoidal structure

$$(\xi^0_0 : J \to I,\ \xi^0 : J \to J \diamond J)$$

on the functor $J : (\mathbb{1},1,1) \to (\mathsf{A},\diamond,I)$

such that the (not explicitly denoted) associativity and unit constraints of the monoidal category $(\mathsf{A},\diamond,I)$ are monoidal natural transformations, *equivalently*, the (not explicitly denoted) associativity and unit constraints of the monoidal category $(\mathsf{A}, \blacklozenge, J)$ are opmonoidal natural transformations.

In [1] this structure was called a *2-monoidal* category. The term *duoidal* appeared in [31].

**Exercise 5.5.** Spell out the diagrams that the morphisms $\xi, \xi^0, \xi_0, \xi^0_0$ of a duoidal category must render commutative.

*Hint.*

The first — so called *associativity* — diagram of Definition 2.5 on the monoidal functor $(\diamond, \xi^0, \xi)$ takes the form

$$\begin{tikzcd}
(X \diamond Y) \blacklozenge (X' \diamond Y') \blacklozenge (X'' \diamond Y'') \arrow[r, "\xi_{X,Y,X',Y'} \blacklozenge 1"] \arrow[d, "1 \blacklozenge \xi_{X',Y',X'',Y''}"'] & ((X \blacklozenge X') \diamond (Y \blacklozenge Y')) \blacklozenge (X'' \diamond Y'') \arrow[d, "\xi_{X \blacklozenge X', Y \blacklozenge Y', X'', Y''}"] \\
(X \diamond Y) \blacklozenge ((X' \blacklozenge X'') \diamond (Y' \blacklozenge Y'')) \arrow[r, "\xi_{X,Y,X' \blacklozenge X'', Y' \blacklozenge Y''}"'] & (X \blacklozenge X' \blacklozenge X'') \diamond (Y \blacklozenge Y' \blacklozenge Y'')
\end{tikzcd}$$

This coincides with the compatibility condition between the opmonoidal associativity constraint of the monoidal category $(\mathsf{A}, \blacklozenge, J)$ and the binary parts of the opmonoidal functors between which it goes.

The second — so called *unitality* — diagram of Definition 2.5 on the monoidal functor $(\diamond, \xi^0, \xi)$ takes the form

$$\begin{tikzcd}
X \diamond Y \arrow[r, "\xi^0 \blacklozenge 1"] \arrow[d, "1 \blacklozenge \xi^0"'] \arrow[dr, equal] & (J \diamond J) \blacklozenge (X \diamond Y) \arrow[d, "\xi_{J,J,X,Y}"] \\
(X \diamond Y) \blacklozenge (J \diamond J) \arrow[r, "\xi_{X,Y,J,J}"'] & X \diamond Y
\end{tikzcd}$$

Its triangles coincide with the compatibility conditions between the opmonoidal left and right unit constraints of the monoidal category $(\mathsf{A}, \blacklozenge, J)$ and the binary parts of the opmonoidal functors between which they go.

The associativity condition on the monoidal functor $(I, \xi_0^0, \xi_0)$ takes the form

$$\begin{tikzcd}
I \blacklozenge I \blacklozenge I \arrow[r, "\xi_0 \blacklozenge 1"] \arrow[d, "1 \blacklozenge \xi_0"'] & I \blacklozenge I \arrow[d, "\xi_0"] \\
I \blacklozenge I \arrow[r, "\xi_0"'] & I
\end{tikzcd}$$

This coincides with the compatibility condition between the opmonoidal associativity constraint of the monoidal category $(\mathsf{A}, \blacklozenge, J)$ and the nullary parts of the opmonoidal functors between which it goes.

The unitality condition on the monoidal functor $(I, \xi_0^0, \xi_0)$ takes the form

$$\begin{tikzcd}
I \arrow[r, "\xi_0^0 \blacklozenge 1"] \arrow[d, "1 \blacklozenge \xi_0^0"'] \arrow[dr, equal] & I \blacklozenge I \arrow[d, "\xi_0"] \\
I \blacklozenge I \arrow[r, "\xi_0"'] & I
\end{tikzcd}$$

Its triangles coincide with the compatibility conditions between the opmonoidal left and right unit constraints of the monoidal category $(\mathsf{A}, \blacklozenge, J)$ and the nullary parts of the opmonoidal functors between which they go.

Next we draw the compatibility diagram between the monoidal associativity constraint of the monoidal category $(\mathsf{A},\diamond,I)$ and the binary parts of the monoidal functors between which it goes:

$$\begin{tikzcd}
(X\diamond Y\diamond Z)\blacklozenge(X'\diamond Y'\diamond Z') \arrow[r, "\xi_{X\diamond Y,Z,X'\diamond Y',Z'}"] \arrow[d, "\xi_{X,Y\diamond Z,X',Y'\diamond Z'}"'] & ((X\diamond Y)\blacklozenge(X'\diamond Y'))\diamond(Z\blacklozenge Z') \arrow[d, "\xi_{X,Y,X',Y'}\diamond 1"] \\
(X\blacklozenge X')\diamond((Y\diamond Z)\blacklozenge(Y'\diamond Z')) \arrow[r, "1\diamond\xi_{Y,Z,Y',Z'}"'] & (X\blacklozenge X')\diamond(Y\blacklozenge Y')\diamond(Z\blacklozenge Z')
\end{tikzcd}$$

This coincides with the coassociativity condition on the opmonoidal functor $(\blacklozenge,\xi_0,\xi)$. Next the compatibility diagram between the monoidal associativity constraint of the monoidal category $(\mathsf{A},\diamond,I)$ and the nullary parts of the monoidal functors between which it goes:

$$\begin{tikzcd}
J \arrow[r, "\xi^0"] \arrow[d, "\xi^0"'] & J\blacklozenge J \arrow[d, "\xi^0\blacklozenge 1"] \\
J\blacklozenge J \arrow[r, "1\blacklozenge\xi^0"'] & J\blacklozenge J\blacklozenge J
\end{tikzcd}$$

This coincides with the coassociativity condition on the opmonoidal functor $(J,\xi_0^0,\xi^0)$. The compatibility diagrams between the monoidal left and right unit constraints of the monoidal category $(\mathsf{A},\diamond,I)$ and the binary parts of the monoidal functors between which they go, are the triangles of

$$\begin{tikzcd}
X\blacklozenge X' \arrow[r, "\xi_{I,X,I,X'}"] \arrow[d, "\xi_{X,I,X',I}"'] \arrow[dr, equal] & (I\blacklozenge I)\diamond(X\blacklozenge X') \arrow[d, "\xi_0\diamond 1"] \\
(X\blacklozenge X')\diamond(I\blacklozenge I) \arrow[r, "1\diamond\xi_0"'] & X\blacklozenge X'
\end{tikzcd}$$

Together they coincide with the counitality conditions on the opmonoidal functor $(\blacklozenge,\xi_0,\xi)$. Finally, the compatibility diagrams between the monoidal left and right unit constraints of the monoidal category $(\mathsf{A},\diamond,I)$ and the nullary parts of the monoidal functors between which they go, are the triangles of

$$\begin{tikzcd}
J \arrow[r, "\xi^0"] \arrow[d, "\xi^0"'] \arrow[dr, equal] & J\diamond J \arrow[d, "\xi_0^0\diamond 1"] \\
J\diamond J \arrow[r, "1\diamond\xi_0^0"'] & J
\end{tikzcd}$$

Together they coincide with the counitality conditions on the opmonoidal functor $(J,\xi_0^0,\xi^0)$.

**Examples 5.6.**

(1) A *braiding* on a monoidal category $(\mathsf{A},\otimes,K)$ is an invertible natural transformation $c$ between $\otimes:\mathsf{A}\times\mathsf{A}\to\mathsf{A}$ and its opposite $\otimes\circ\mathsf{flip}:\mathsf{A}\times\mathsf{A}\to\mathsf{A}$ — with components $c_{X,Y}:X\otimes Y\to Y\otimes X$ for arbitrary objects $X,Y$ of $\mathsf{A}$ — such that the following

diagrams commute for all objects $X,Y$ and $Z$.

$$\begin{tikzcd}
X\otimes Y\otimes Z \arrow[rr, "c_{X,Y\otimes Z}"] \arrow[dr, "c_{X,Y}\otimes 1"'] & & Y\otimes Z\otimes X \\
& Y\otimes X\otimes Z \arrow[ur, "1\otimes c_{X,Z}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& X\otimes Z\otimes Y \arrow[dr, "c_{X,Z}\otimes 1"] & \\
X\otimes Y\otimes Z \arrow[ur, "1\otimes c_{Y,Z}"] \arrow[rr, "c_{X\otimes Y,Z}"'] & & Z\otimes X\otimes Y
\end{tikzcd}$$

A braiding $c$ is called a *symmetry* if $c_{Y,X}\circ c_{X,Y}$ is the identity morphism $X\otimes Y\to X\otimes Y$ for all objects $X$ and $Y$.

The monoidal category of sets in part (1) of Examples 2.2, and the monoidal category of vector spaces in part (2) of Examples 2.2 are symmetric monoidal categories with symmetry provided by the flip maps. However, the monoidal category of bimodules in part (3) of Examples 2.2, and the monoidal category of endofunctors in part (4) of Examples 2.2 are not even braided in general.

Any braided monoidal category $(\mathsf{A},\otimes,K,c)$ can be regarded as a duoidal category with equal monoidal structures $(\diamond,I)=(\otimes,K)=(\blacklozenge,J)$ and

$$\xi_0^0:\ K\xrightarrow{1}K \qquad \xi_0:\ K\otimes K\xrightarrow{\cong}K$$
$$\xi^0:\ K\xrightarrow{\cong}K\otimes K \qquad \xi:\ X\otimes Y\otimes V\otimes Z\xrightarrow{1\otimes c_{Y,V}\otimes 1}X\otimes V\otimes Y\otimes Z.$$

(2) For an arbitrary set $X$, there is a category $\mathsf{span}(X)$ whose objects are triples consisting of a set $A$ and two maps $s,t:A\to X$. Such an object can be visualized as a directed graph with vertex set $X$ and edge set $A$; the maps $s$ and $t$ taking the edges to their source and target, respectively. The morphisms are maps $f:A\to A'$ which are compatible with the maps to $X$ in the sense of the following commutative diagram.

$$\begin{tikzcd}
& A \arrow[dl, "t"'] \arrow[dr, "s"] \arrow[dd, "f"] & \\
X & & X \\
& A' \arrow[ul, "t'"] \arrow[ur, "s'"'] &
\end{tikzcd}$$

This category $\mathsf{span}(X)$ admits the following duoidal structure. The first monoidal product is

$$X \xleftarrow{(p,q)\mapsto t'(p)=t(q)} A'\diamond A:=\{(p,q)\in A'\times A | t'(p)=t(q),\ s'(p)=s(q)\} \xrightarrow{(p,q)\mapsto s'(p)=s(q)} X$$

— that is, the set of pairs of "parallel edges" in $A$ and $A'$ — with monoidal unit $X\xleftarrow{(x,y)\mapsto x}X\times X\xrightarrow{(x,y)\mapsto y}X$ . The other monoidal product is

$$X \xleftarrow{(p,q)\mapsto t'(p)} A'\blacklozenge A:=\{(p,q)\in A'\times A | s'(p)=t(q)\} \xrightarrow{(p,q)\mapsto s(q)} X$$

— that is, the set of pairs of "consecutive edges" in $A$ and $A'$ — with monoidal unit $X \xleftarrow{1} X \xrightarrow{1} X$ . The structure morphisms are the following.

$$
\begin{array}{llrl}
\xi_0^0 : X \to X \times X, & & x \mapsto & (x,x) \\
\xi_0 : (X \times X) \blacklozenge (X \times X) \to X \times X, & & (x, y = x', y') \mapsto & (x, y') \\
\xi^0 : X \to X \cong X \diamond X, & & x \mapsto & x \\
\xi : (A \diamond B) \blacklozenge (A' \diamond B') \to (A \blacklozenge A') \diamond (B \blacklozenge B'), & & (a,b),(a',b') \mapsto & ((a,a'),(b,b'))
\end{array}
$$

in picture: $\begin{pmatrix} \xleftarrow{a} \\ \xleftarrow{b} \end{pmatrix} \begin{pmatrix} \xleftarrow{a'} \\ \xleftarrow{b'} \end{pmatrix} \overset{\xi}{\mapsto} \begin{matrix} (\xleftarrow{a} \ \xleftarrow{a'}) \\ (\xleftarrow{b} \ \xleftarrow{b'}) \end{matrix}$

(3) For a separable Frobenius algebra $(B, i, \delta)$, the category of $B^{\mathrm{e}}$-bimodules admits the following duoidal structure. The monoidal product $V \blacklozenge W$ of $B^{\mathrm{e}}$-bimodules $V$ and $W$ is their usual $B^{\mathrm{e}}$-module tensor product

$$V \otimes_{B^{\mathrm{e}}} W := V \otimes W / \{ v \cdot (b \otimes b') \otimes w - v \otimes (b \otimes b') \cdot w \},$$

whose monoidal unit is $B^{\mathrm{e}}$ with actions provided by the multiplication. The other monoidal product $V \diamond W$ is a twisted $B^{\mathrm{e}}$-module tensor product

$$V \otimes W / \{ (b \otimes 1) \cdot v \cdot (b' \otimes 1) \otimes w - v \otimes (1 \otimes i(1_{\langle 1 \rangle} b) 1_{\langle 2 \rangle}) \cdot w \cdot (1 \otimes b') \}$$

(where $1_{\langle 1 \rangle} \otimes 1_{\langle 2 \rangle} = \delta(1)$ and implicit summation is understood). The monoidal unit for this product is $B^{\mathrm{e}}$ with suitably twisted actions. The structure morphisms $\xi_0^0, \xi^0, \xi_0, \xi$ are given in terms of the separable Frobenius algebra structure $(i, \delta)$ of $B$, see [8].

**Proposition 5.7.** *Consider a duoidal category* $(\mathsf{A}, \diamond, I, \blacklozenge, J)$.

*(1) For any comonoid* $(C, d, e)$ *in the monoidal category* $(\mathsf{A}, \diamond, I)$*, there is an opmonoidal functor* $C \blacklozenge -$ *on* $(\mathsf{A}, \diamond, I)$ *whose binary part has the component*

$$C \blacklozenge (X \diamond Y) \xrightarrow{d \blacklozenge (1 \diamond 1)} (C \diamond C) \blacklozenge (X \diamond Y) \xrightarrow{\xi_{C,C,X,Y}} (C \blacklozenge X) \diamond (C \blacklozenge Y)$$

*when evaluated at any objects X and Y, and the nullary part is* $C \blacklozenge I \xrightarrow{e \blacklozenge 1} I \blacklozenge I \xrightarrow{\xi_0} I$ .

*(2) Any comonoid morphism* $f : (C, d, e) \to (C', d', e')$ *in the monoidal category* $(\mathsf{A}, \diamond, I)$ *induces an opmonoidal natural transformation* $C \blacklozenge - \to C' \blacklozenge -$ *between the opmonoidal functors of part (1); with components* $f \blacklozenge 1 : C \blacklozenge X \to C' \blacklozenge X$ *when evaluated at any object* $X$.

*Proof.* (1) Coassociativity and counitality of the comonad $(C, d, e)$, and opmonoidality of the functor $(\blacklozenge, \xi_0, \xi)$ together imply the opmonoidality of the functor $C \blacklozenge -$ with the stated binary and nullary parts.

(2) is immediate by the naturality of $\xi$. □

In the opposite direction, we can get a comonoid from any opmonoidal functor $t : (\mathsf{A}, \diamond, I) \to (\mathsf{A}, \diamond, I)$ in the following way. The monoidal unit $I$ is a comonoid as in part (5) of Examples 5.2. Hence the opmonoidal functor $t$ takes it to a comonoid $tI$ as in part (6) of

Examples 5.2. Let us stress, however, that this is *not the inverse* of the construction in Proposition 5.7, of an opmonoidal structure on the functor $C\blacklozenge -$ from a comonoid structure on $C$; it is *not a bijection* in general. In the earlier examples of bimodule categories its bijectivity essentially depended on the generator properties of the monoidal unit.

**Corollary 5.8.** *Consider a duoidal category* $(\mathsf{A},\diamond,I,\blacklozenge,J)$.

*(1) $J$ is a comonoid in the monoidal category $(\mathsf{A},\diamond,I)$ with comultiplication $\xi^0$ and counit $\xi^0_0$.*

*(2) For any comonoids $(C,d,e)$ and $(C',d',e')$ in the monoidal category $(\mathsf{A},\diamond,I)$, there is a comonoid $C\blacklozenge C'$ in $(\mathsf{A},\diamond,I)$ with comultiplication and counit*

$$C\blacklozenge C' \xrightarrow{d\blacklozenge d'} (C\diamond C)\blacklozenge(C'\diamond C') \xrightarrow{\xi_{C,C,C',C'}} (C\blacklozenge C')\diamond(C\blacklozenge C') \quad \textit{and} \quad C\blacklozenge C' \xrightarrow{e\blacklozenge e'} I\blacklozenge I \xrightarrow{\xi_0} I.$$

*Proof.* (1) The only object of the singleton category $\mathbb{1}$ is a comonoid by part (5) of Examples 5.2. Hence the opmonoidal functor $(J,\xi^0_0,\xi^0)$ takes it to the comonoid in $(\mathsf{A},\diamond,I)$, see part (6) of Examples 5.2. The resulting comonoid $J$ is that in the claim.

(2) The functor $C'\blacklozenge - : (\mathsf{A},\diamond,I)\to(\mathsf{A},\diamond,I)$ is opmonoidal by part (1) of Proposition 5.7. Hence it takes the comonoid $(C,d,e)$ to a comonoid by part (6) of Examples 5.2. The resulting comonoid $C'\blacklozenge C$ is that in the claim. □

Symmetrical arguments prove that $I$, as well as the $\diamond$-product of any monoids in $(\mathsf{A},\blacklozenge,J)$, are monoids in $(\mathsf{A},\blacklozenge,J)$.

**Definition 5.9.** A *bimonoid* in a duoidal category $\mathsf{A}$ consists of

- an object $T$ of $\mathsf{A}$
- a monoid structure $(m:T\blacklozenge T\to T, u:J\to T)$ in $(\mathsf{A},\blacklozenge,J)$
- a comonoid structure $(d:T\to T\diamond T, e:T\to I)$ in $(\mathsf{A},\diamond,I)$

such that $d$ and $e$ are monoid morphisms in $(\mathsf{A},\blacklozenge,J)$, equivalently, $m$ and $u$ are comonoid morphisms in $(\mathsf{A},\diamond,I)$, equivalently, the following diagrams commute.

$$\begin{tikzcd}
T\blacklozenge T \arrow[r, "m"] \arrow[d, "d\blacklozenge d"'] & T \arrow[r, "d"] & T\diamond T \\
(T\diamond T)\blacklozenge(T\diamond T) \arrow[rr, "\xi_{T,T,T,T}"'] & & (T\blacklozenge T)\diamond(T\blacklozenge T) \arrow[u, "m\diamond m"']
\end{tikzcd}
\qquad
\begin{tikzcd}
T\blacklozenge T \arrow[r, "m"] \arrow[d, "e\blacklozenge e"'] & T \arrow[d, "e"] \\
I\blacklozenge I \arrow[r, "\xi_0"'] & I
\end{tikzcd}$$

$$\begin{tikzcd}
J \arrow[r, "u"] \arrow[d, "\xi^0"'] & T \arrow[d, "d"] \\
J\diamond J \arrow[r, "u\diamond u"'] & T\diamond T
\end{tikzcd}
\qquad
\begin{tikzcd}
J \arrow[r, "u"] \arrow[dr, "\xi^0_0"'] & T \arrow[d, "e"] \\
& I
\end{tikzcd}$$

**Examples 5.10.**

(1) Regard a braided monoidal category $(\mathsf{A},\otimes,K,c)$ as a duoidal category, as described in part (1) of Examples 5.6. A bimonoid in this duoidal category reduces to the usual notion of bimonoid in the braided monoidal category $(\mathsf{A},\otimes,K,c)$: it consists of
- an object $T$ of $\mathsf{A}$
- a monoid structure $(m:T\otimes T\to T, u:K\to T)$ in $(\mathsf{A},\otimes,K)$
- a comonoid structure $(d:T\to T\otimes T, e:T\to e)$ in $(\mathsf{A},\otimes,K)$

such that the following diagrams commute.

$$\begin{tikzcd}
T\otimes T \arrow[r, "m"] \arrow[d, "d\otimes d"'] & T \arrow[r, "d"] & T\otimes T \\
T\otimes T\otimes T\otimes T \arrow[rr, "1\otimes c_{T,T}\otimes 1"'] & & T\otimes T\otimes T\otimes T \arrow[u, "m\otimes m"']
\end{tikzcd}
\qquad
\begin{tikzcd}
T\otimes T \arrow[r, "m"] \arrow[d, "e\otimes e"'] & T \arrow[d, "e"] \\
K\otimes K \arrow[r, "\cong"'] & K
\end{tikzcd}$$

$$\begin{tikzcd}
K \arrow[r, "u"] \arrow[d, "\cong"'] & T \arrow[d, "d"] \\
K\otimes K \arrow[r, "u\otimes u"'] & T\otimes T
\end{tikzcd}
\qquad
\begin{tikzcd}
K \arrow[r, "u"] \arrow[dr, equal] & T \arrow[d, "e"] \\
& K
\end{tikzcd}$$

(2) Let us take next the duoidal category $\mathsf{span}(X)$ of part (2) of Examples 5.6 for an arbitrary set $X$. First we describe the comonoids in $(\mathsf{span}(X),\diamond,X\times X)$. From an arbitrary object $(A,t,s)$ there is a unique map of spans to $X\times X$, the one sending $a$ to $(t(a),s(a))$. Hence there is a unique counital comultiplication $A\to A\diamond A$: the diagonal one sending $a$ to $(a,a)$. Moreover, any morphism in $\mathsf{span}(X)$ is clearly a morphism of comonoids. This amounts to saying that a bimonoid in $\mathsf{span}(X)$ is the same thing as a monoid therein. A monoid, on the other hand, is precisely a category with the object set $X$.

(3) Take next a separable Frobenius algebra $B$ and the duoidal category of $B^{\mathsf{e}}$-bimodules in part (3) of Examples 5.6. It was proven in [8] that the bimonoids therein are precisely the weak bialgebras whose base algebra is $B$. The proof is much too technical even to sketch here.

**Theorem 5.11.** *Any bimonoid $(T,m,u,d,e)$ in a duoidal category $(\mathsf{A},\diamond,I,\blacklozenge,J)$ determines a bimonad on $(\mathsf{A},\diamond,I)$ as follows.*

- *the underlying functor is $T\blacklozenge - : \mathsf{A}\to\mathsf{A}$*
- *when evaluated at an arbitrary object $X$, the multiplication of the monad has the component $m\blacklozenge 1 : T\blacklozenge T\blacklozenge X\to T\blacklozenge X$, and the unit has the component $u\blacklozenge 1 : X\to T\blacklozenge X$*
- *when evaluated at arbitrary objects $X$ and $Y$, the binary part of the opmonoidal structure has the component*

$$T\blacklozenge(X\diamond Y)\xrightarrow{d\blacklozenge(1\diamond 1)}(T\diamond T)\blacklozenge(X\diamond Y)\xrightarrow{\xi_{T,T,X,Y}}(T\blacklozenge X)\diamond(T\blacklozenge Y)$$

*and the nullary part is* $T\blacklozenge I\xrightarrow{e\blacklozenge 1}I\blacklozenge I\xrightarrow{\xi_0}I$

*This bimonad is a Hopf monad if and only if the following natural transformation is invertible.*

$$T\blacklozenge(X\diamond(T\blacklozenge Y))\xrightarrow{d\blacklozenge 1}(T\diamond T)\blacklozenge(X\diamond(T\blacklozenge Y))\xrightarrow{\xi_{T,T,X,T\blacklozenge Y}}(T\blacklozenge X)\diamond(T\blacklozenge T\blacklozenge Y)\xrightarrow{1\diamond(m\blacklozenge 1)}(T\blacklozenge X)\diamond(T\blacklozenge Y)$$

*Proof.* By the functoriality of the monoidal product $\blacklozenge$, both $m\blacklozenge 1$ and $u\blacklozenge 1$ are natural transformations. By the associativity and unitality of the monoid $(T,m,u)$ the associativity and unitality axioms of a monad hold (see Exercise 5.3).

Opmonoidality of the functor $T\blacklozenge -$ with the stated binary and nullary parts follows by part (1) of Proposition 5.7. The multiplication and the unit of the monad $T\blacklozenge -$ are opmonoidal natural transformations by part (2) of Proposition 5.7.

The final claim about $T\blacklozenge -$ being a Hopf monad follows by a straightforward substitution. □

Let us stress again that this construction in Theorem 5.11, of a bimonad structure on the functor $T\blacklozenge -$ from a bimonoid structure on $T$ is *not a bijection* in general. In the earlier examples of bimodule categories its bijectivity essentially depended on the generator properties of the monoidal unit.

**Examples 5.12.**

(1) Regard a braided monoidal category $(\mathsf{A},\otimes,K,c)$ as a duoidal category, as in part (1) of Example 5.6, and take a bimonoid $(T,m,u,d,e)$ therein. The induced bimonad $T\otimes -$ on $(\mathsf{A},\otimes,K)$ turns out to be a Hopf monad if and only if the morphism

$$\beta:\ T\otimes T \xrightarrow{d\otimes 1} T\otimes T\otimes T \xrightarrow{1\otimes m} T\otimes T$$

is invertible. Similarly to Theorem & Example 3.3, this is equivalent to the existence of an *antipode* morphism $s: T\to T$ for which the following diagram commutes.

$$\begin{tikzcd}
T \arrow[r, "d"] \arrow[d, "d"'] \arrow[dr, "e"] & T\otimes T \arrow[r, "1\otimes s"] & T\otimes T \arrow[d, "m"] \\
T\otimes T \arrow[r, "s\otimes 1"'] & K \arrow[r, phantom] & A \\
& T\otimes T \arrow[ur, "m"']
\end{tikzcd}$$

$$\begin{array}{ccccc}
T & \xrightarrow{d} & T\otimes T & \xrightarrow{1\otimes s} & T\otimes T \\
{\scriptstyle d}\downarrow & \searrow^{e} & & & \downarrow{\scriptstyle m} \\
& & K & \xrightarrow{u} & A \\
T\otimes T & \xrightarrow[s\otimes 1]{} & T\otimes T & \xrightarrow[m]{} & A
\end{array}$$

Indeed, if $\beta$ is invertible then $s$ is constructed as the composite morphism

$$T \xrightarrow{1\otimes u} T\otimes T \xrightarrow{\beta^{-1}} T\otimes T \xrightarrow{e\otimes 1} T.$$

Conversely, if there is an antipode morphism $s$ then $\beta$ has the inverse

$$T\otimes T \xrightarrow{d\otimes 1} T\otimes T\otimes T \xrightarrow{1\otimes s\otimes 1} T\otimes T\otimes T \xrightarrow{1\otimes m} T\otimes T.$$

A bimonoid equipped with a (necessarily unique) antipode is called a *Hopf monoid*.

(2) Take next the duoidal category $\mathsf{span}(X)$ of part (2) of Examples 5.6 for an arbitrary set $X$. By part (2) of Examples 5.10, a bimonoid therein is a category $\mathsf{A}$ with object set $X$. The induced bimonad $\mathsf{A}\blacklozenge -$ on $(\mathsf{span}(X),\diamond,X\times X)$ is a Hopf monad if and only if the map

$$A\diamond A\to\{(c,c')\in\mathsf{A}\times\mathsf{A}|t(c)=t(c')\},\qquad (c,c')\mapsto(c,c\circ c')$$

is invertible. This happens to be the case if and only if $\mathsf{A}$ is a groupoid; that is, every morphism in $\mathsf{A}$ is invertible; see [8]. The map sending a morphism of $\mathsf{A}$ to its inverse plays the role of an antipode map.

(3) Take finally the duoidal category $\mathsf{bim}(B^{\mathsf{e}})$ of bimodules over the enveloping algebra $B^{\mathsf{e}}$ of a separable Frobenius algebra $B$, discussed in part (3) of Example 5.6. Recall from part (3) of Examples 5.10 that a bimonoid therein is a weak bialgebra $T$ with base algebra $B$. The induced bimonad $T\blacklozenge -$ on $(\mathsf{bim}(B^{\mathsf{e}}),\diamond,I)$ is a Hopf monad if and only if the map of (4.12) is invertible, equivalently (see Theorem & Definition 4.5), $T$ is a weak Hopf algebra.

**5.13. About the antipode.** In all of the cases discussed in Examples 5.12 we saw that the bimonad induced by a bimonoid is a Hopf monad if and only if some kind of antipode exist. We should stress that this is *not a general feature* of bimonoids in a duoidal category; in general the induced bimonad can be a Hopf monad without having an antipode behind.

In fact, the existence of certain antipode morphisms in the examples of Examples 5.12 follows from the fact that all of them belong to a distinguished class of bimonoids in duoidal categories, discussed in [9].

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